# Final Solutions

15-122 Principles of Imperative Computation

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Recitation Section: S

## Instructions

- This exam is closed-book with one sheet of notes permitted.
- You have 180 minutes to complete the exam.
- There are 7 problems on 26 pages (including 2 blank pages and 2 reference sheets at the end).
- Use a **dark** pen or pencil to write your answers.
- Read each problem carefully before attempting to solve it.
- Do not spend too much time on any one problem.
- Consider if you might want to skip a problem on a first pass and return to it later.

| | Max | Score |
|---|---|---|
| Contracts [C0 $\mapsto$ C] | 30 | |
| Safety [C] | 20 | |
| Complexity | 20 | |
| Mergeable Heaps [C] | 30 | |
| Virtual Machines [C] | 60 | |
| Graph Representation [C] | 50 | |
| Shortest Path [C] | 40 | |
| Total: | 250 | |

# 1 Contracts [C0 $\mapsto$ C] (30 points)

The following C0 code checks the validity of a union-find data structure implemented as an array. Recall that each element in the array maps a vertex in a graph to its canonical representative. It does not do height tracking. *You do not need to understand the details of this code.*

Recall that the array utilities `le_seg(x,A,lo,hi)` and `gt_seg(x,A,lo,hi)` implement the tests $x \leq A[lo, hi)$ and $x > A[lo, hi)$, respectively.

```
1  bool no_loops(int[] UFS, int n, int v)
2  //@requires n == \length(UFS);
3  //@requires 0 <= v && v < n;
4  //@requires le_seg(0, UFS, 0, v);
5  //@requires gt_seg(n, UFS, 0, v);
6  {
7    int w = v;  // Starting point
8    int k = 0;  // Counter
9    while (UFS[w] != w && k < n)
10   //@loop_invariant 0 <= k && k <= n;
11   {
12     w = UFS[w];
13     k++;
14   }
15   return k < n;
16 }
17
```

```
18 bool is_ufs(int[] UFS, int n) {
19   if (!is_array_expected_length(UFS, n))
20     return false;
21
22   for (int v = 0; v < n; v++)
23   //@loop_invariant 0 <= v && v <= n;
24   //@loop_invariant le_seg(0, UFS, 0, v);
25   //@loop_invariant gt_seg(n, UFS, 0, v);
26   {
27     if (!(0 <= UFS[v] && UFS[v] < n))
28       return false;
29
30     if (!no_loops(UFS, n, v))
31       return false;
32   }
33   return true;
34 }
```

5pts **Task 1** Prove that the loop invariant on line 25 is true initially (INIT).

To show: `gt_seg(n, UFS, 0, 0)`

Because

$x > A[y, y)$ for every $x$ and $y$

10pts **Task 2** Prove that the loop invariant on line 25 is preserved by each iteration of the loop (PRES). *You may not need all lines.*

Assume: `gt_seg(n, UFS, 0, v)`

To show: `gt_seg(n, UFS, 0, v')`

| | | | |
|---|---|---|---|
| (a) | `UFS[v] < n` | by | line 27 |
| (b) | `gt_seg(n, UFS, 0, v)` | by | assumption |
| (c) | `gt_seg(n, UFS, 0, v+1)` | by | (a) and (b) |
| (d) | `v' = v+1` | by | line 22 |
| (e) | | by | |

5pts **Task 3** Is the array access on line 9 safe? If you think it is, provide a proof. If you think it is not, give a 3-element array `UFS` and a values for `v` so that the call `no_loops(UFS, 3, v)` causes a memory violation.

☐ Safe,
because _____

☒ Unsafe
Counterexample: `v ==` ___2___, `UFS ==`

| 0 | 1 | 2 |
|---|---|---|
| 1 | 2 | 42 |

10pts **Task 4** Translate the function `no_loops` into C. You may assume that the array utility functions have been correctly ported to C for you. If a C0 contract cannot be expressed in C, briefly explain why in a comment. Note in particular that you will need to translate the loop invariant on line 10 into appropriately placed `ASSERT` statements.

```
bool no_loops(int *UFS, int n, int v)
//@requires n == \length(UFS);  // NOT SUPPORTED IN C
{
  REQUIRES(0 <= v && v <= n);
  REQUIRES(le_seg(0, UFS, 0, v) && gt_seg(n, UFS, 0, v));

  int w = v;  // Starting point
  int k = 0;  // Counter
  ASSERT(0 <= k && k < n);
  while (UFS[w] != w && k < n) {
    w = UFS[w];
    k++;
    ASSERT(0 <= k && k <= n);
  }
  return k < n;
}
```

## 2 Safety [C] (20 points)

For the C functions in this question, write (additional) preconditions that are sufficient to ensure that there will be **no undefined behavior and no memory leaks**. Your preconditions must be as permissive as possible: they should allow the function to run *whenever* undefined behavior and memory leaks would not occur. Make sure it's not possible to cause undefined behavior in the precondition itself!

If it's not possible to ensure that a function is free of undefined behavior and memory leaks, then write the precondition `false`, which indicates that the function cannot be run safely. If no preconditions are needed, write the precondition `true`.

Do not assume *any* implementation-defined behaviors except that a byte is 8 bits. Your preconditions should make your functions safe for any implementation-defined behavior.

4pts **Task 1**

```
unsigned short a1(unsigned short x, int z) {

    REQUIRES(          0 <= z && z < 8*sizeof(short)          );

    return x << z; // (Also accept 0 < z && z < 8*sizeof(short))
}
```

4pts **Task 2**

```
 void a2(long **A, size_t n) {

     REQUIRES(                    false                    );

     for (size_t i = 0; i < n; i++)
       if (A[i] != NULL) free(A[i]);
     free(A);
 }
```

4pts **Task 3**

```
int a3(unsigned short n) {

    REQUIRES(                    true                    );

    if (n == 0) return -1;

    int x = 0;
    int *B = xcalloc(n, sizeof(int));
    for (size_t i = 0; i < n; i++)
        x += B[i];
    free(B);
    return x;
}
```

4pts **Task 4** For this task, assume that `A` is a pointer to a heap-allocated array of size `n`.

```
void a4(int *A, size_t n) {

    REQUIRES(_____________________________ n <= 1 _____________________________);

    for (size_t i = 0; i < n; i++)
      free(&A[i]);
}
```

4pts **Task 5** For this task, assume that casting between signed and unsigned types of the same size works as seen in class.

```
int a5(signed char n, int m) {

    REQUIRES(_____________________ n >= 0 || m >= 0 _____________________);

    unsigned int x = (unsigned int)(unsigned char)n;
    unsigned int y = (unsigned int)(signed int)n;

    if (x != y) return INT_MIN + m;  // overflow if m < 0
    return m;                        // x != y when n < 0
}
```

## 3 Complexity (20 points)

Give **worst-case** running time bounds for the following operations. Some of the descriptions state that multiple operations are happening: give the cost of doing the *entire sequence* of operations, not the cost of doing a single operation within the sequence.

If the worst case depends on amortized behavior, also write the word **AMORTIZED**.

Assume comparison functions and hash functions are all $O(1)$.

Always give the *simplest*, *tightest* big-$O$ bound.

| | **Worst-case** |
|---|---|
| Adding $n$ elements to an initially empty stack (implemented as a linked list). | $O(n)$ |
| Adding $n$ elements to an unbounded array, using **mergesort** to re-sort the array after **every single addition**. | $O(n^2 \log n)$ **AMORTIZED** |
| Adding $n$ elements to an unbounded array, using **quicksort** to re-sort the array **once, after all** $n$ **additions**. | $O(n^2)$ (**AMORTIZED**) |
| Inserting an element into a min-heap that already has $n$ elements in it and then removing the minimum from the min-heap. | $O(\log n)$ |
| Adding $n$ elements to an initially empty binary search tree. | $O(n^2)$ |
| Adding $n$ elements to an initially empty AVL tree. | $O(n \log n)$ |
| Inserting $n$ elements to an initially empty non-resizing, separate-chaining hash table with table size $m$. | $O(n^2)$ |
| Computing the number of chains that are empty in a separate-chaining hash table with $n$ elements and table size $m$. | $O(m)$ |
| Running Kruskal's algorithm on a graph with $v$ vertices and $e$ edges using depth-first search to verify if two vertices are already connected. | $O(e^2)$ or $O(ev^2)$ |
| Running Kruskal's algorithm on a graph with $v$ vertices and $e$ edges using union-find with height tracking to minimize tree height. | $O(e \log e)$ |

# 4 Mergeable Heaps [C] (30 points)

This question explores *mergeable heaps*. Rather than growing a heap by adding an element to it, mergeable heaps are grown by merging two existing heaps. Viewed as trees, mergeable heaps maintain the same *order invariant* as the min-heaps discussed in class, but may impose *different* invariants on the heap's *shape*.

**Task 1** We start out with a *naive* version of a mergeable heap, which only maintains the order invariant but does not constrain the heap's shape. Below is a summary of the relevant type definitions and C0-style prototypes. For simplicity, we record the priority as the field `value` rather than as we saw in class.

```
typedef struct heap_header heap;
struct heap_header {
  int value; // Node priority
  heap *left;
  heap *right;
};

bool empty(heap *H);
//@requires is_mergeable_heap(H);
```

```
heap* merge(heap *L, heap *R)
//@requires is_mergeable_heap(L);
//@requires is_mergeable_heap(R);
//@ensures is_mergeable_heap(\result);

heap* delete_min(heap *H);
//@requires is_mergeable_heap(H);
//@requires !empty(H);
//@ensures is_mergeable_heap(\result);
```

4pts

**a.** Complete the specification function `is_minheap` in the box below, defining the *order invariant*, which is the same as the one for the min-heaps discussed in class. The specification function is called from within the representation invariant `is_mergeable_heap`:

```
bool is_mergeable_heap(heap *H) { // Don't worry about circularity
  if (H == NULL) return true;
  return is_minheap(H, H->value);
}
```

*Hint: you will not need additional `NULL`-checks, nor additional space.*

```
bool is_minheap(heap *H, int lower_bound) {
  if (H == NULL) return true;

  int prio = H->value;
  return prio >= lower_bound
      && is_minheap(H->left, prio)
      && is_minheap(H->right, prio);

}
```

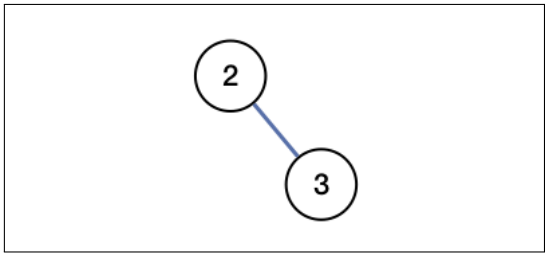
To maintain the order invariant, the function `merge(L, R)` merges heaps `L` and `R` by considering the priority $p_L$ of the root $r_L$ of `L` and the priority $p_R$ of the root $r_R$ of `R`:

- if $p_L \leq p_R$, it chooses $r_L$ to be the root of the resulting tree and merges the right subtree of `L` with `R`.
- if $p_L > p_R$, it chooses $r_R$ to be the root of the resulting tree and merges the right subtree of `R` with `L`.

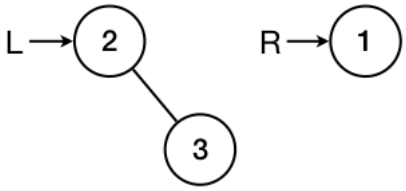
In summary, `merge` chooses the heap with the smaller root and merges its right subtree with the other heap. Note that `merge` leaves the left subtree of the chosen heap intact.

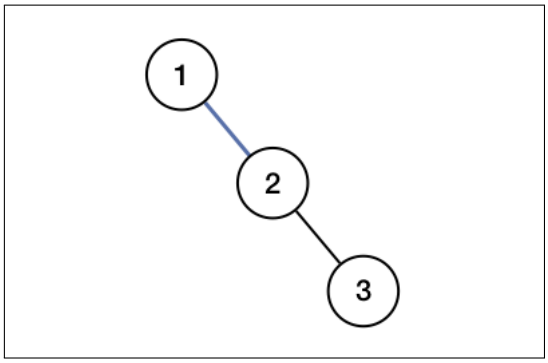
For each of the examples below, draw the heap resulting from calling the function `merge` on the two given heaps `L` and `R` in the box provided. Each node contains its priority.

2pts **b.**

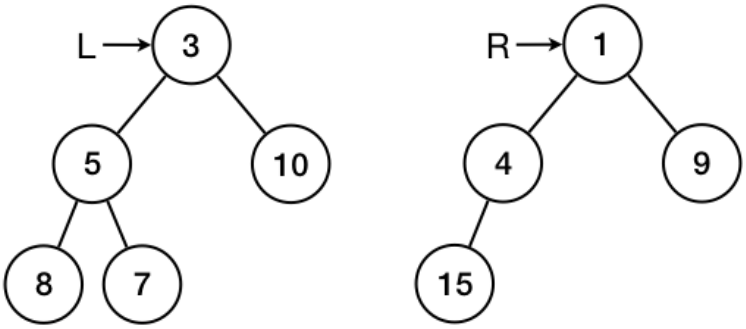



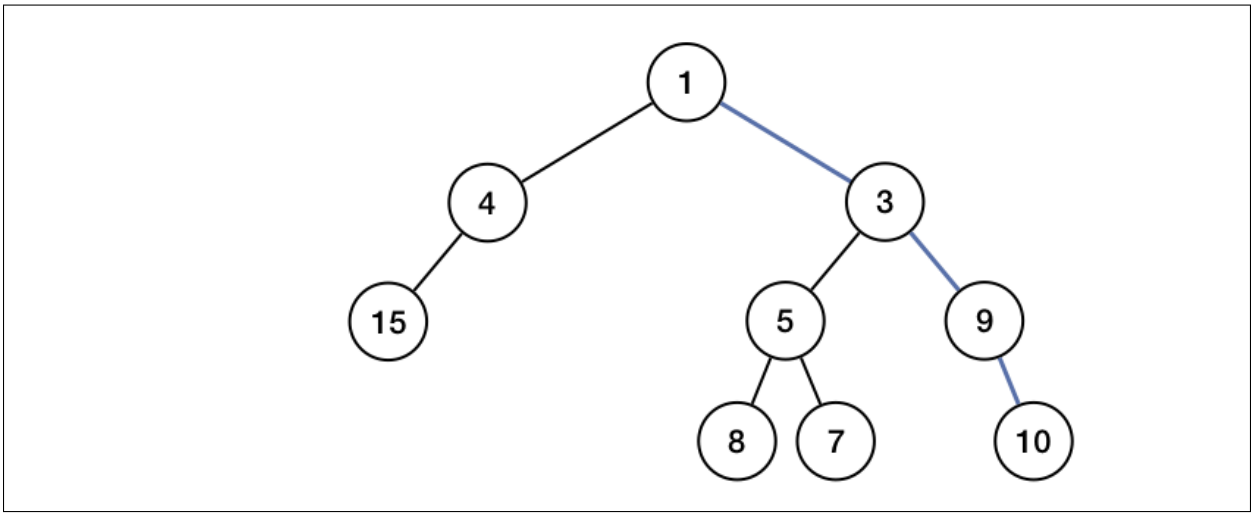

2pts **c.**




3pts **d.**



8pts **e.** Complete the function `merge`, which implements the specification provided in Task 1.b. You can use the type definitions and prototypes given on page 7:

```
heap* merge(heap *L, heap *R) {
  REQUIRES(is_mergeable_heap(L) && is_mergeable_heap(R));

  if (L == NULL) ____________________ return R ____________________;

  if (R == NULL) ____________________ return L ____________________;

  if (L->value <= R->value) {

    L->right = _____________ merge(L->right, R) _____________;

    ENSURES(_____________ is_mergeable_heap(L) _____________);

    return _____________ L _____________;
  } else {
    R->right = _____________ merge(R->right, L) _____________;

    ENSURES(_____________ is_mergeable_heap(R) _____________);

    return _____________ R _____________;
  }
}
```

4pts **f.** Complete function `delete_min`, which returns the heap resulting from removing the root from the heap `H`. You can use the type definitions and prototypes given on page 7:

```
heap* delete_min(heap *H) {
  REQUIRES(_____ is_mergeable_heap(H) && !empty(H) _____);

  _____ heap *res = merge(H->left, H->right) _____;

  _____ free(H) _____;

  ENSURES(_____ is_mergeable_heap(res) _____);

  _____ return res _____;
}
```

2pts **g.** What is the simplest and tightest worst-case complexity of `merge`, when invoked on heaps with $n_L$ and $n_R$ nodes? *Hint: observe that* `merge` *only traverses right subtrees.*

$O($ $n_L + n_R$ $)$

**Task 2** Next, we touch on *leftist heaps*, a more elaborate form of mergeable heaps, that maintain in addition to the order invariant also the *leftist shape invariant*. The shape invariant makes sure that the left subtree of each node is at least as deep as its right subtree.

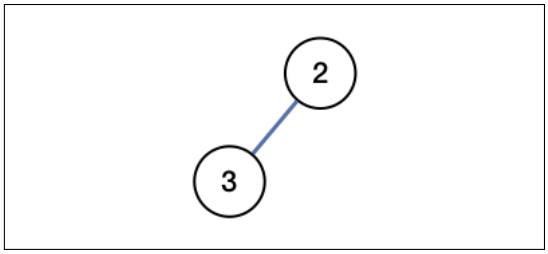
To express this invariant, we extend the heap struct given in Task 1 with the field `rank`, which records the number of nodes along the path from the root of a heap to its rightmost leaf. More formally, the rank of a heap $H$ is:

- 0 if $H$ is empty (i.e., `H == NULL`), and otherwise
- 1 plus the rank of its right subtree.

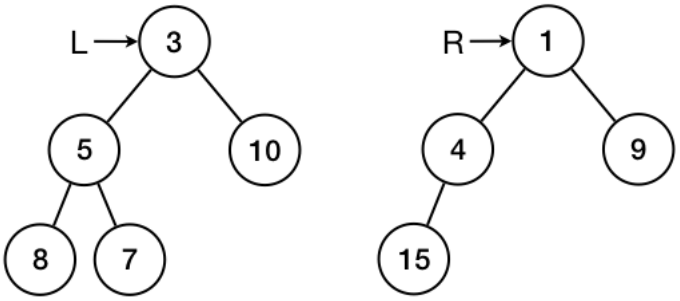
A heap $H$ is *leftist*, if $H$ satisfies the order invariant and if, for every node $x$ in $H$, the rank of $x$'s left subtree is *greater than or equal to* the rank of $x$'s right subtree.

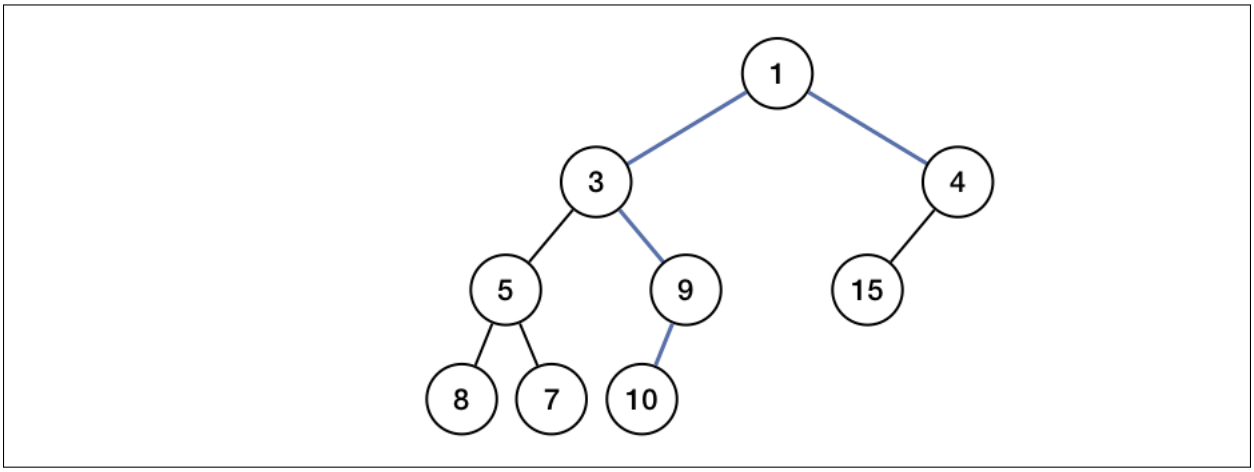
To maintain the leftist shape invariant, `merge` now needs to swap the left subtree with the right subtree before returning, if the rank of the left subtree is smaller than the rank of the right subtree. For each of the examples below, draw the heap resulting from calling the new `merge` on the two given heaps `L` and `R` in the box provided:

2pts **a.**


3pts **b.** You may find the scratch pages at the end of this exam handy.



## 5 Virtual Machines [C] (60 points)

This question deals with the C0VM. The specifications of a selection of C0VM instructions can be found on page 25 of this exam. You may assume the same implementation-defined C behaviors encountered in homework (e.g., 8-bit **`char`**s and 4-byte **`int`**s).

**Task 1** The C0VM instruction `ildc` loads large integer constants, for example `0x15122`, onto the operand stack. In this exercise, we will pretend that this instruction does not exist, and achieve the same effect with some of the remaining instructions on page 25. We will do so in two steps.

4pts

**a.** Write a C0 expression that evaluates to `0x15122` using only integers in the range $[-128, 128)$. Feel free to express these integers in either decimal or hexadecimal at your convenience. *Hint: C0's bitwise operators may come really handy.*

```
(0x01 << 16) | (0x51 << 8) | 0x22  == 0x15122
```

Alternatives: `(((0x01 << 8)|0x51) << 8)|0x22`; `2*((0x51 << 8)|0x22)`

10pts

**b.** Using the above as a guideline, write a C0VM bytecode fragment that loads the integer `0x15122` onto the operand stack. Write the bytecode values in the left column (e.g., "`15 0A`"). Feel free to write the corresponding instructions (here "`vload 10`"), and the contents of the stack in comments to their right.

```
                                    # Initial stack: S

10 01          # bipush 0x01

10 10          # bipush 16

78             # ishl

10 51          # bipush 0x51

10 08          # bipush 8

78             # ishl

80             # ior

10 22          # bipush 0x22

80             # ior

                                    # Final stack: S, 0x15122
```

**Task 2** It is sometimes useful to provide special bytecode instructions for operations that programmers use very frequently. One such operation is the increment or decrement of a value by a small integer, e.g., `i+2` or `x-1`.

We will do so using the new bytecode operation `iincr <b>`, which increments the (integer) value on the top of the operand stack by `<b>`, a signed number that fits in one byte. If `<b>` is negative, this value is decremented. Using the notation seen in class, it is defined as follows:

```
0xd0 iincr <b>    S, x:w32 -> S, x+b:w32
```

10pts

**a.** Implement the instruction `iincr` according to the above spec. Recall that `val2int` converts a `c0_value` to an integer, that `int2val` converts an integer to a `c0_value`, that `c0v_pop` pops a `c0_value` from the operand stack and that `c0v_push` pushes a `c0_value` onto the stack. You can assume that `P` is a `ubyte` array that holds the bytecode, `pc` is the current program counter, `S` is the operand stack, and `V` is the `c0_value` array containing the local variables of the current function. Recall also that this is part of a **`switch`** statement. Make sure to cast appropriately.

```
case IINCR: {
  pc++;
  int32_t b = (byte)P[pc];
  pc++;
  int32_t x = val2int(c0v_pop(S));
  c0v_push(S, int2val(x+b));
  break;
}
```

8pts

**b.** Using `iincr`, complete the bytecode that implements the function

```
int f(int x) {
  x++;
  return x;
}
```

You will need exactly the number of lines provided. Fill all blanks.

```
#<f>
00 01            # number of arguments = 1
00 01            # number of local variables = 1

00 09            # code length =   9   bytes

15 00            # vload 0        # x

D0 01            # iincr 1        # (x + 1)

36 00            # vstore 0       # x += 1

15 00            # vload 0        # x
B0               # return         #
```

**Task 3** This simple example suggests an instruction even more useful than `iincr`: an instruction that increments a *local variable* rather than the value at the top of the stack. This will allow C0 statements like `x++;` or `i -= 3;` to be implemented using a single C0VM instruction.

2pts **a.** Give an abstract definition of this instruction, which we call `vincr`:

```
0xd1 vincr <b,i>          S -> S          (V[i] = V[i] + b)
```

10pts **b.** Complete the implementation of this instruction:

```
case VINCR: {
  pc++;
  int32_t b = (byte)P[pc];
  pc++;
  size_t i = P[pc];
  pc++;
  V[i] = int2val(val2int(V[i]) + b);
  break;
}
```

4pts **c.** Complete the bytecode for the function `f` in Task 2.b using `vincr` rather than `iincr`. Fill all blanks.

```
#<f>
00 01           # number of arguments = 1
00 01           # number of local variables = 1

00 06           # code length =   6   bytes

D1 01 00        # vincr 1         # (x + 1)

15 00           # vload 0         # x
B0              # return          #
```

**Task 4** The compiler has developed a bug and is producing bytecode that may cause assertion failures or undefined behavior in your VM! (It's not too bad, though: the examples below have no funny business with the code length, and the mnemonics reported in comments faithfully represent the bytecode to the left.)

If any of these main functions execute without error or incident, write down the value they will return. If any of these main functions causes assertion failures or undefined behavior in the VM, then circle the *first* line that will cause unacceptable (e.g., assertion failures), undefined, or unpredictable behavior when executed by the C0VM.

4pts **a.**

```
00 00        # number of arguments = 0
00 00        # number of local variables = 0
00 08        # code length = 8 bytes
10 05        # bipush 5
10 FF        # bipush -1
59           # dup
68           # imul
78           # ishl
B0           # return       <----- 10
```

4pts **b.**

```
00 00        # number of arguments = 0
00 00        # number of local variables = 0
00 08        # code length = 8 bytes
13 00 01     # ildc 15122 (defined in integer pool)
10 0A        # bipush 10
64           # isub
60           # iadd         <----- Not enough args on stack
B0           # return
```

4pts **c.**

```
00 00       # number of arguments = 0
00 00       # number of local variables = 0
00 14       # code length = 20 bytes
10 15       # bipush 21
59          # dup
9F 00 06    # if_cmpeq +6
A7 00 0F    # goto +15    <----- Offset goes beyond end of code
10 00       # bipush 0
10 2A       # bipush 42
4E          # imstore     <----- Pointer expected, found integer
A7 00 05    # goto +5
10 2A       # bipush 42
B0          # return
```

## 6 Graph Representation [C] (50 points)

The minimal graph interface seen in class, and reported below, did not provide a function that returns the neighbors of a vertex. As a consequence, some of the algorithms we developed, like DFS and BFS, had a higher complexity than otherwise achievable. In this exercise, we will add such a function, `get_neighbors`, together with some building blocks. The resulting extended interface is as follows (extensions are marked as `//NEW`):

```
typedef unsigned int vertex;
typedef struct graph_header* graph_t;

typedef struct adjlist_node adjlist;                    // NEW
struct adjlist_node {                                   // NEW
  vertex vert;                                          // NEW
  adjlist *next;                                        // NEW
};                                                      // NEW

graph_t graph_new(unsigned int numvert);
//@ensures \result != NULL;

void graph_free(graph_t G);
//@requires G != NULL;

unsigned int graph_size(graph_t G);
//@requires G != NULL;

bool graph_hasedge(graph_t G, vertex v, vertex w);
//@requires G != NULL;
//@requires v < graph_size(G) && w < graph_size(G);

void graph_addedge(graph_t G, vertex v, vertex w);
//@requires G != NULL;
//@requires v < graph_size(G) && w < graph_size(G);
//@requires v != w && !graph_hasedge(G, v, w);

adjlist* get_neighbors(graph_t G, vertex v);            // NEW
//@requires G != NULL && v < graph_size(G);             // NEW

void free_neighbors(adjlist *L);                        // NEW
```

The type `adjlist`, taken identically from the adjacency list representation of graphs, is simply a `NULL`-terminated linked list of vertices. The function `get_neighbors(G, v)` returns the neighbors of vertex `v` in graph `G` as such a list. It should not allocate new memory unless it really has to. The function `free_neighbors(L)` disposes of a list of vertices `L` returned by `get_neighbors`, *if appropriate*.

**Task 1** The following client-side code uses the new functions to print a graph.

```
1  void graph_print(graph_t G) {
2    REQUIRES(G != NULL);
3    for (vertex v = 0; v < graph_size(G); v++) {
4      printf("Vertices connected to %u: ", v);
5      for (adjlist *p = get_neighbors(G, v); p != NULL; p = p->next)
6        printf(" %u,", p->vert);
7      printf("\n");
8      free_neighbors(p);
9    }
10 }
```

3pts **a.** The above code has an error. On what line does this error occur and what is causing it?

Line 8: the variable `p` is out of scope; also it might free `NULL`.

7pts **b.** Modify the code for `print_graph` above in order to fix this error (you may abbreviate the print statements if you want).

```
void graph_print(graph_t G) {
  REQUIRES(G != NULL);

  for (vertex v = 0; v < graph_size(G); v++) {
    printf("Vertices connected to %u: ", v);
    adjlist *nbors = get_neighbors(G, v);            // Fix
    for (adjlist *p = nbors; p != NULL; p = p->next) // Updated
      printf(" %u,", p->vert);
    printf("\n");
    free_neighbors(nbors);                           // Updated
  }

}
```

**Task 2** We will now extend the *adjacency list* implementation of the graph interface with the added functions. Recall the definition of the internal type `graph` and the prototype of some of the specification functions:

```
typedef struct graph_header graph;
struct graph_header {
  unsigned int size;
  adjlist **adj;
};
```

```
bool is_vertex(graph *G, vertex v);
bool is_graph(graph *G);
```

6pts **a.** *Efficiently* implement the function `get_neighbors`. Include appropriate contracts.

```
adjlist* get_neighbors(graph *G, vertex v) {

  REQUIRES(is_graph(G) && is_vertex(G, v));
  return G->adj[v];

}
```

3pts **b.** What is the complexity of `get_neighbors` as a function of the number $v$ of vertices and the number $e$ of edges in the input graph?

$O($ 1 $)$

3pts **c.** Complete the body of `free_neighbors` so that, by the time we are done using a graph, *all allocated memory has been freed* and *none has been freed twice*.

```
void free_neighbors(adjlist *L) {

  (void)L;   // OPTIONAL: just to make the compiler happy
  return;

}
```

3pts **d.** Using this implementation of graphs, what is the simplest and tightest worst-case complexity of `graph_print`? *Hint: consider how many things get printed.*

$O($ $\max(e, v)$ $)$

**Task 3** Next, we will do the same with the *adjacency matrix* implementation of the graph interface. The internal type `graph` is now defined as follows:

```
typedef struct graph_header graph;
struct graph_header {
  unsigned int size;
  bool *matrix; // 1-D array
};
```

```
bool is_vertex(graph *G, vertex v);
bool is_graph(graph *G);
```

In the code questions below, recall that you can use interface functions.

6pts **a.** *Efficiently* implement the function `get_neighbors`. Include appropriate contracts.

```
adjlist* get_neighbors(graph *G, vertex v) {

  REQUIRES(is_graph(G) && is_vertex(G, v));
  adjlist *L = NULL;
  for (vertex w = 0; w < graph_size(G); w++) {
    if (graph_hasedge(G, v, w)) {
      adjlist *node = xmalloc(sizeof(adjlist));
      node->vert = w;
      node->next = L;
      L = node;
    }
  }
  return L;

}
```

3pts **b.** What is the complexity of `get_neighbors` as a function of the number $v$ of vertices and the number $e$ of edges in the input graph?

$O($ $v$ $)$

3pts **c.** Complete the body of `free_neighbors` so that, by the time we are done using a graph, *all allocated memory has been freed* and *none has been freed twice*.

```
void free_neighbors(adjlist *L) {

  if (L == NULL) return;
  free_neighbors(L->next);
  free(L);

}
```

3pts **d.** Using this implementation of graphs, what is the simplest and tightest worst-case complexity of `graph_print`? *Hint: consider how many things get printed.*

$O($ $\max(e, v^2)$ $)$

**Task 4** The adjacency matrix code you worked on in lab implemented the matrix as an array of arrays (a 2D array). The above type **`struct`** `graph_header` uses instead a one-dimensional array, like in the images assignment. This part asks you to write some additional functions relative to one-dimensional arrays.

5pts **a.** Complete the specification function `is_graph(G)` that checks that the adjacency matrix representation of its input graph is valid. The helper function `is_valid_row` checks that the row for vertex `v` of the adjacency matrix of graph `G` is valid.

```
bool is_valid_row(graph *G, vertex v) {
  REQUIRES(G != NULL);

  size_t v_row = ____v * G->size____; // Start index of v's row

  if (____G->matrix[v_row + v] == true____) // No self loops
    return false;

  for (____vertex w = 0; w < G->size; w++____) {
    // Every outgoing edge a corresponding edge coming back to it

    if (__G->matrix[v_row + w] != G->matrix[w * G->size + v]__)
      return false;
  }
  return true;
}

bool is_graph(graph *G) {

  if (G == NULL) return ____false____;

  for (vertex v = 0; v < G->size; v++)
  {
    if (!is_valid_row(G, v)) return ____false____;
  }
  return ____true____;
}
```

5pts **b.** Complete the function `graph_new`, including the appropriate contracts.

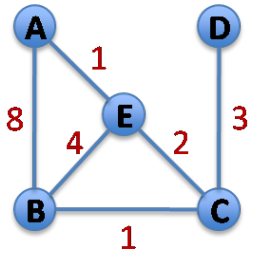
```
graph* graph_new(unsigned int size) {

  graph *G = xmalloc(sizeof(graph));
  G->size = size;
  G->matrix = xcalloc(size*size, sizeof(bool));
  ENSURES(is_graph(G));
  return G;

}
```

## 7 Shortest Path [C] (40 points)

In a graph with weighted edges, the *weight of a path* is the sum of the weights of its edges. For example, the path ($A$-$B$-$C$-$D$) in the example graph on the right has weight 12 ($= 8 + 1 + 3$). The path ($A$-$B$) has weight 8 since its one edge has weight 8. The empty path which starts and ends at vertex $A$ has weight 0.




A *shortest path* between two vertices $v$ and $w$ is a path of minimum weight among all the paths between them. Its weight is called the *shortest distance* between $v$ and $w$. In our example, the shortest path between $A$ and $B$ is $A$-$E$-$C$-$B$, and therefore the shortest distance between them is 4.

In this exercise, we will implement a function that computes the shortest distance between a given vertex and every other vertex in a graph. The algorithm it embodies is known as *Dijkstra's algorithm*, after the Dutch computer scientist Edsger Dijkstra.

But first, we need to update the graph interface **studied in class** to handle *weighted* graphs:

```
typedef unsigned int vertex;
typedef struct graph_header* graph_t;

graph_t graph_new(unsigned int numvert); // New graph with numvert vertices

void graph_free(graph_t G);

unsigned int graph_size(graph_t G);  // Number of vertices in the graph

// Returns the weight of (v1,v2) or -1 if they are not connected
int graph_hasedge(graph_t G, vertex v1, vertex v2);
  //@requires v1 < graph_size(G) && v2 < graph_size(G);

void graph_addedge(graph_t G, vertex v1, vertex v2, int weight);
  //@requires v1 < graph_size(G) && v2 < graph_size(G);
  //@requires v1 != v2 && graph_hasedge(G, v1, v2) == -1 && weight > 0;
```

The main changes are highlighted:

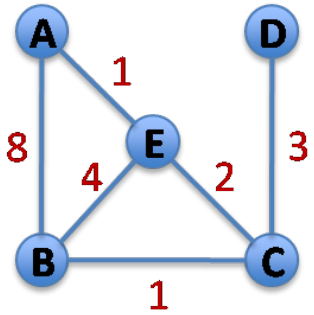
1. If graph `G` has an edge between vertices `v1` and `v2`, the function `graph_hasedge(G,v1,v2)` now returns the weight of this edge. If `v1` and `v2` are not connected, it returns `-1`.
2. Naturally, the function `graph_addedge` now takes the weight of the added edge as an additional argument. Weights are strictly positive.

Throughout this exercise, we will assume for simplicity that the weight of every path in the given graphs is less than `INT_MAX`.

Dijkstra's algorithm takes as input a graph `G` and a vertex `start`. It outputs an array `dist` that maps every vertex in `G` to its shortest distance to `start`. If `start` and `v` are not connected, `dist[v]` contains `INT_MAX`. Like DFS and BFS, Dijkstra's algorithm uses a mark array to keep track of visited vertices. Like these algorithms, it explores the graph `G` beginning at `start` and following the edges of connected vertices. It keeps track of the vertices to visit next in a *priority queue* where the priority is given by the shortest distance between `start` and each vertex at the time of insertion (shorter paths may be discovered later). Shorter distances have higher priority. During execution `dist[v]` contains the shortest distance between `start` and `v` known *so far* (or `INT_MAX` if `v` has not yet been visited).

Dijkstra's algorithm proceeds as follows:

1. Mark all vertices as unvisited.
2. Create `dist` and set the distance to all vertices other than `start` to `INT_MAX`. Set the distance to `start` to 0.
3. Create a priority queue `Q` and insert the pair $(\texttt{start}, 0)$ in it.
4. For as long as `Q` is not empty
   (a) Get the vertex $v$ with minimum weight from `Q`
   (b) If $v$ is unmarked
   - mark it
   - for every unmarked neighbor $w$ of $v$
     i. let $d$ be the weight of the edge $(v, w)$ in `G`
     ii. if `dist[`$v$`] + `$d$` < dist[`$w$`]`
     - update `dist[`$w$`]` to `dist[`$v$`] + `$d$
     - insert $(w, \texttt{dist[}w\texttt{]})$ into `Q`
5. Return `dist`




10pts **Task 1** Complete the table below by simulating the execution of Dijkstra's algorithm on the above graph. The first three rows have been filled for you already. The first row shows the contents of `Q` and `dist`, and the marked vertices just before entering the loop (indicated with $\longrightarrow$ above). The subsequent rows show these values and the selected vertex `v` at the end of each iteration of the loop (indicated as $\Longrightarrow$). `INT_MAX` is abbreviated as $\infty$.

| | v | Q | dist | | | | | marked |
|---|---|---|---|---|---|---|---|---|
| | | | *A* | *B* | *C* | *D* | *E* | |
| $\longrightarrow$ | | $(A,0)$ | 0 | $\infty$ | $\infty$ | $\infty$ | $\infty$ | |
| $\Longrightarrow$ | $A$ | $(B,8),(E,1)$ | 0 | 8 | $\infty$ | $\infty$ | 1 | $A$ |
| $\Longrightarrow$ | $E$ | $(B,8),(B,5),(C,3)$ | 0 | 5 | 3 | $\infty$ | 1 | $A,E$ |
| $\Longrightarrow$ | $C$ | $(B,8),(B,5),(B,4),(D,6)$ | 0 | 4 | 3 | 6 | 1 | $A,C,E$ |
| $\Longrightarrow$ | $B$ | $(B,8),(B,5),(D,6)$ | 0 | 4 | 3 | 6 | 1 | $A,B,C,E$ |
| $\Longrightarrow$ | $B$ | $(B,8),(D,6)$ | 0 | 4 | 3 | 6 | 1 | $A,B,C,E$ |
| $\Longrightarrow$ | $D$ | $(B,8)$ | 0 | 4 | 3 | 6 | 1 | $A,B,C,D,E$ |
| $\Longrightarrow$ | $B$ | | 0 | 4 | 3 | 6 | 1 | $A,B,C,D,E$ |

Since Dijkstra's algorithm relies on a priority queue, we need to define the type of its elements and the notion of priority. The interface of generic priority queues is given on page 26 of this exam. Since we are now working in C, it upgrades the interface we saw in class with functions to deallocate memory. The differences are highlighted.

The type of elements to insert in the priority queue are vertices together with their best-known distance from `start` at that point. This is captured by the type:

```
typedef struct vert_dist_pair vert_dist;
struct vert_dist_pair {
  vertex vert;
  int    dist;  // Best known distance from start to vert
};
```

The (pre-implemented) helper function

```
vert_dist* mk_vert_dist(vertex v, int d);
```

packages vertex `v` and distance `d` into an entity of type `vert_dist` and returns a pointer to it.

6pts **Task 2** Define the function `has_shorter_distance(e1, e2)` so that it returns `true` if the distance component of `e1` is strictly less than the one of `e2`, and `false` otherwise.

```
bool has_shorter_distance(elem e1, elem e2) {
  REQUIRES(e1 != NULL && e2 != NULL);

  vert_dist *vd1 = (vert_dist*)e1;
  vert_dist *vd2 = (vert_dist*)e2;
  return vd1->dist < vd2->dist;

}
```

5pts **Task 3** Assuming our priority queue is not self-resizing, how big should we make it to be sure we won't run out of space no matter how many edges the graph contains? Said differently, what capacity should we specify when calling `pq_new`? Give the smallest such value relative to the number of vertices in the graph.

`graph_size(G) * (graph_size(G) - 1) / 2` or $v(v-1)/2$

4pts **Task 4** Next, we will implement Dijkstra's algorithm as a single non-recursive function. What kind of arrays can the mark array, `mark`, and the shortest distance array, `dist`, be? Check all that apply.

| | | |
|---|---|---|
| `mark`: | ☒ Stack-allocated | ☒ Heap-allocated |
| `dist`: | ☐ Stack-allocated | ☒ Heap-allocated |

15pts **Task 5** Based on the pseudo-code on page 21 (and mostly repeated below), complete the following client-side function that implements Dijkstra's algorithm.

```
int* dijkstra(graph_t G, vertex start) {
  REQUIRES(G != NULL && start < graph_size(G));

  // Mark all vertices as unvisited

      bool     mark[graph_size(G)]                                    ;

  for (vertex v = 0; v < graph_size(G); v++)
    mark[v] = false;    // v has not been visited yet

  // Create dist and set the distance to all vertices other than
  // start to INT_MAX.  Set the  distance to start to 0.

      int *    dist = xmalloc(graph_size(G) * sizeof(int))           ;

  for (vertex v = 0; v < graph_size(G); v++)
    dist[v] = INT_MAX;  // No known distance from start
  dist[start] = 0;

  // Create a priority queue Q and insert (start,0) in it.
  int capacity = /* Your answer from task 3 -- nothing to do */;

  pq_t Q = pq_new(capacity, &has_shorter_distance,      &free      );

  pq_add(Q,                  mk_vert_dist(start, 0)                 );
```

*(Continued on next page)*

**Task 6** *(continued from previous page)*

```
  while (        !pq_empty(Q)         ) // For as long as Q is not empty
  {
    // Get the vertex v with minimum weight from Q

    vert_dist *vd = ____________________ pq_rem(Q) ____________________;

    vertex v = ____________________ vd->vert ____________________;

    if (_____________ !mark[v] _____________) { // if v is unmarked

      _____________ mark[v] = true _____________; // mark it

      // for every unmarked neighbor w of v,
      //    let d be the weight of the edge (v,w)
      for (vertex w = 0; w < graph_size(G); w++) {

        int d = _____________ graph_hasedge(G, v, w) _____________;

        if (_____________ d == -1 && !mark[w] _____________) {

          if (dist[v] + d < dist[w]) {

            _____ dist[w] = dist[v] + d _____; // update dist[w]

            // insert (w, dist[w]) into Q

            _____ pq_add(Q, mk_vert_dist(w, dist[w])) _____;
          }
        }
      }
    }
    free(vd); ____________________ // anything to clean up?
  }
  pq_free(Q); ____________________ // anything to clean up?

  ____________________ // anything to clean up?
  return dist;
}
```

## REFERENCE: Selected C0VM Instructions

```
Stack operations
0x59 dup          S, v -> S, v, v
0x57 pop          S, v -> S

Arithmetic
0x60 iadd         S, x:w32, y:w32 -> S, x+y:w32
0x7E iand         S, x:w32, y:w32 -> S, x&y:w32
0x6C idiv         S, x:w32, y:w32 -> S, x/y:w32
0x68 imul         S, x:w32, y:w32 -> S, x*y:w32
0x80 ior          S, x:w32, y:w32 -> S, x|y:w32
0x70 irem         S, x:w32, y:w32 -> S, x%y:w32
0x78 ishl         S, x:w32, y:w32 -> S, x<<y:w32
0x7A ishr         S, x:w32, y:w32 -> S, x>>y:w32
0x64 isub         S, x:w32, y:w32 -> S, x-y:w32
0x82 ixor         S, x:w32, y:w32 -> S, x^y:w32

Local Variables
0x15 vload <i>    S -> S, v                    (v = V[i])
0x36 vstore <i>   S, v -> S                    (V[i] = v)

Constants
0x01 aconst_null   S -> S, null:*
0x10 bipush <b>    S -> S, x:w32               (x = (w32)b, sign extended)
0x13 ildc <c1,c2>  S -> S, x:w32               (x = int_pool[(c1<<8)|c2])
0x14 aldc <c1,c2>  S -> S, a:*                 (a = &string_pool[(c1<<8)|c2])

Control Flow
0x00 nop               S -> S
0x9F if_cmpeq <o1,o2>  S, v1, v2 -> S          (pc = pc+(o1<<8|o2) if v1 == v2)
0xA0 if_cmpne <o1,o2>  S, v1, v2 -> S          (pc = pc+(o1<<8|o2) if v1 != v2)
0xA1 if_icmplt <o1,o2> S, x:w32, y:w32 -> S    (pc = pc+(o1<<8|o2) if x < y)
0xA7 goto <o1,o2>      S -> S                  (pc = pc+(o1<<8|o2))
0xB0 return            ., v -> .               (return v to caller)

Memory
0xBB new <s>   S -> S, a:*                     (*a is now allocated, size <s>)
0x2E imload    S, a:* -> S, x:w32              (x = *a, a != NULL, load 4 bytes)
0x4E imstore   S, a:*, x:w32 -> S              (*a = x, a != NULL, store 4 bytes)
```

## REFERENCE: Generic Priority Queues in C

```
/*************************** Client interface ***************************/

typedef void* elem;  // Supplied by client

// f(x,y) returns true if e1 is STRICTLY higher priority than e2
typedef bool has_higher_priority_fn(elem e1, elem e2);
//@requires e1 != NULL && e2 != NULL;

typedef void elem_free_fn(elem e);
//@requires e != NULL;


/*************************** Library interface ***************************/

typedef struct heap_header* pq_t;

bool pq_empty(pq_t Q)
  /*@requires Q != NULL; @*/ ;

bool pq_full(pq_t Q)
  /*@requires Q != NULL; @*/ ;

pq_t pq_new(int capacity, has_higher_priority_fn* prior, elem_free_fn* fr)
  /*@requires capacity > 0 && prior != NULL && fr != NULL; @*/
  /*@ensures \result != NULL && pq_empty(\result); @*/ ;

void pq_add(pq_t Q, elem x)
  /*@requires Q != NULL && !pq_full(Q) && x != NULL; @*/ ;

elem pq_rem(pq_t Q)
  /*@requires Q != NULL && !pq_empty(Q); @*/
  /*@ensures \result != NULL; @*/ ;

elem pq_peek(pq_t Q)
  /*@requires Q != NULL && !pq_empty(Q); @*/
  /*@ensures \result != NULL; @*/ ;

void pq_free(pq_t Q)
  /*@requires Q != NULL; @*/ ;
```

C1-style contracts are given as documentation. Changes w.r.t. in-class version are highlighted.