Human Population Growth: Stability or Explosion?

An historical survey of various models of population growth which gives attention to their character, derivations, and flaws.

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1. Deterministic population models

In this paper we will examine several simple, but historically interesting, models for human population growth, all of which are special cases of the following general population model. We suppose that the size N of the population under consideration is a function of time t, and that its rate of change dN/dt is a "known" function of N and t. (In practice, the rate of change is usually *conjectured* rather than known, and the resulting differential equation is studied with regard to the reasonableness of its consequences.) To be a little more precise, we suppose that the population has a birth rate (per individual) B and a mortality rate M, both of which may be functions of N and t, expressed as fractions of the total population, so that the *net* growth rate may be written as

$$\frac{dN}{dt} = BN - MN = (B - M)N = PN, \tag{1}$$

where $P = B - M$ is the **production** rate, that is, the net rate at which individuals in the population are reproducing.

One notices immediately that there is something wrong here: N is an *integer*-valued function, so it doesn't make much sense for it to have a nonzero derivative at every point, as suggested by equation (1).

For biological populations with discrete generations (e.g., cicadas), it is more appropriate to use a difference equation for the growth rate, for example,

$$\frac{\Delta N}{\Delta t} = \frac{N(t + \Delta t) - N(t)}{\Delta t} = PN,$$

where Δt is the length of a generation. However, large human populations vary "almost continuously" (births and deaths are taking place all the time), so it is reasonable to *model* the true N by letting the discrete generation time Δt tend to zero. This yields equation (1).

A model such as equation (1), expressed solely in terms of differential equations that ignore random fluctuations in the population, is called **deterministic.** Under modest assumptions about the niceness of the function P, equation (1) together with a single known population size, say $N = N_0$ at time $t = t_0$, uniquely determines N at any future time [**22**, Section 56]. Since we know that biological populations are subject to many random effects (ranging from chance encounters of lovers or enemies to natural disasters such as earthquakes), any single deterministic model will be at best a crude approximation, and then possibly only for a limited time or place. Nevertheless, even the simplest such models may contribute to our understanding of changing phenomena, as we shall see.

2. The Malthusian or exponential model

About the turn of the 19th century, the British economist Thomas Malthus (1766–1834) observed that biological populations (including human ones) tended to increase at a rate proportional to the population size **[15]**. In terms of our general model (1) this is equivalent to the assumption that the production rate P is constant, which of course would be the case if both birth and mortality rates were constant. One learns in freshman calculus (see any of **[8]**, **[9]**, **[24]**, **[25]**, or **[26]**) how to solve equation (1) with P constant by "separating the variables" to obtain

$$N = N_0 e^{Pt}, \tag{2}$$

where N_0 is the population at time $t = 0$. Thus the assumption of constant production rate leads to the conclusion that the population must grow exponentially. Malthus reasoned that such an exponential growth of the world's population could not go on indefinitely, and therefore some sort of catastrophe (war, plague, or famine, for example) must intervene from time to time to interrupt the inexorable working of equation (1) by artificially (and rapidly) reducing N.

The Malthusian model leaves a lot to be desired as a description of human population growth, since it takes into account hardly any of the important characteristics of human reproductive behavior — nothing that would distinguish us from laboratory colonies of bacteria, say. The other models to be considered here are similarly unsophisticated, and differ only from the Malthusian model by changing the assumption that the production rate is constant.

3. The Verhulst or logistic model

In the 1840's, a Belgian mathematician, P. F. Verhulst (1804–1849), proposed the following alternative to the Malthusian model (rediscovered and popularized in the 1920's by Pearl and Reed **[17]**). Assume the birth rate B is constant, but the mortality rate M is proportional to the population, say $M = mN$ for some constant m. Then the growth model (1) becomes

$$\frac{dN}{dt} = (B - mN)N. \tag{3}$$

Such a mortality rate might result, for example, from competition for available resources, such as living space, food supply, air, water, etc. Using available census data from the period 1790–1840 to determine the constants B and m, Verhulst predicted the population of the United States in 1940 and was off by less than one percent. Of course, the success of the prediction was the result of the averaging out of several important short-term fluctuations in population (caused by World War I and the depression, for example), and as it turned out, 1940 was a lucky choice for a stopping point. (The original Verhulst model also had a small, positive, constant term on the right-hand side of (3) to account for net immigration to the U.S. Such a term complicates the mathematical analysis without adding any enlightenment about population models. Furthermore, our objective is to compare simple models for *world* population growth, which rules out immigration, as far as we know.)

Equation (3) raises a possibility that does not occur with the Malthusian model, namely that there might be a non-zero **equilibrium** population, one for which $N' = 0$. Specifically, this would be the case if $N = B/m$. The possibility of a non-zero equilibrium population is very important, as we shall see, so we introduce the abbreviation $\lambda = B/m$ for this special number. Observe from equation (3) that, if $0 < N < \lambda$, then $N' > 0$, so the population is increasing, and if $N > \lambda$, then $N' < 0$, so the population is decreasing. This is an example of a **stable** equilibrium: If the population is not *at* equilibrium, it is moving *toward* it.

By factoring out the birth rate B, the Verhulst model may be rewritten as

$$\frac{dN}{dt} = B\left(1 - \frac{N}{\lambda}\right)N. \tag{4}$$

The factor $1 - N/\lambda$ may be interpreted as an "environmental resistance" factor, with λ interpreted as the "maximum supportable population". When N is small compared to λ, the resistance factor is close to 1, and the model resembles the Malthusian case. However, when N becomes sufficiently large, the resistance factor causes the growth rate to tend to zero and the population to stabilize.

Equation (4) may be differentiated implicitly to obtain

$$N'' = B\left(1 - \frac{2N}{\lambda}\right)N', \tag{5}$$

from which we see that there is an inflection point at $N = \lambda/2$, and the growth rate is maximal when the population reaches half its maximum supportable level. For $N < \lambda/2$, the graph of N is concave upward, resembling the exponential growth curve, but for $N > \lambda/2$, the graph is concave downward and leveling off. This S-shape is called the **logistic** growth curve. It has been quite successful as a model for laboratory populations (for example, bacteria or fruit flies) with limited resources such as space or food supply, and in at least one case, as noted above, for describing the gross changes in a large human population over a century. However, we know from many examples that the Verhulst model does not adequately describe either short-range changes or very long-range trends in human population growth, whether for a given geographical region or for the earth as a whole. Pearl and Reed [**17**] predicted a *maximum* world population of about two billion, which was exceeded by 1930.

We have by now discovered the important qualitative features of the solution of (4) without actually solving the equation. Its solution is not difficult to obtain, however, by separation of variables:

$$N = \frac{\lambda N_0}{N_0 + (\lambda - N_0)e^{-Bt}}. \tag{6}$$

The details may be found in [**8**], [**9**], or [**24**], or worked out as an exercise. The integration step provides a good reason for studying partial fraction decompositions.

4. The von Foerster or Doomsday model

A little less than a generation ago, H. von Foerster, P. M. Mora, and L.W. Amiot published an article [**27**] entitled "Doomsday: Friday, 13 November, A.D. 2026", which suggested quite a different approach to the problem of formulating a gross model for world population growth. They argued from a historical perspective that as human population has increased, improvements in technology and in mass communication have had the effect of welding the population into a more and more effective "coalition" in a vast "game against nature", rendering natural environmental hazards less effective, improving living conditions, and extending the average life span. It follows (they said) that the net production rate P might actually be an *increasing* function of N rather than a decreasing one. It should, of course, be a very slowly increasing function, and they tentatively proposed

$$P = P_0 N^{1/k}, \tag{7}$$

for some constants P_0 and k to be determined from historical data. This leads to the differential equation

$$\frac{dN}{dt} = P_0 N^{1+1/k}. \tag{8}$$

As with the previous models, equation (8) may be solved by separation of variables. In this case the integration step requires nothing deeper than the power rule, and the resulting equation is easily solved for N:

$$N = \frac{k^k}{(C - P_0 t)^k} \tag{9}$$

where C is a constant of integration.

Now equation (9) leads to a very disturbing conclusion: There is a finite time t, namely $t = C/P_0$, at which N becomes infinite, hence the phrase "population explosion". That is a far more striking prediction of future catastrophe than even Malthus could come up with. Hence it is important to know whether there is any possibility that such a model could accurately reflect world population trends, and if so, how far off is "Doomsday". If we abbreviate C/P_0 as t_0, then (9) may be simplified to

$$N = \frac{K}{(t_0 - t)^k}, \tag{10}$$

for appropriate constants K, k, t_0. The question of whether (10), and hence (8), is plausible as a model may be resolved by "fitting" the equation to available historical data to determine the "best" values for K, k, and t_0, and then observing whether the "best fit" is indeed a good fit. This was done by von Foerster and his colleagues, using the least squares method to fit equation (10) to all of the independent estimates of world population they could find, ranging over the span from 0 to 1958, when their article was written. The best values of the parameters were found to be:

$$\begin{aligned} t_0 &= \text{A.D. } 2026.87 \pm 5.50 \text{ years}, \\ K &= (1.79 \pm 0.14) \times 10^{11}, \\ k &= 0.990 \pm 0.009. \end{aligned} \tag{11}$$

In particular, t_0 is the Doomsday of their title. The root mean square (rms) error of the fit was 7%; this is a very good fit, considering the uncertainties and inconsistencies of estimates of world population prior to 1900.

If we take logs on both sides of (10) we have

$$\ln N = \ln K - k \ln(t_0 - t); \tag{12}$$

so $\ln N$ is a linear function of $\ln(t_0 - t)$ with negative slope $-k$. Thus a further test of the plausibility of the model is to make a double logarithmic plot of the data and see whether they appear to lie on a straight line. Specifically, $\ln N$ is to be plotted against $\ln \tau$, where $\tau = t_0 - t$ is "doomstime", the time left before the population grows without bound. The results of such a plot are shown in FIGURE 1, adapted from [27].

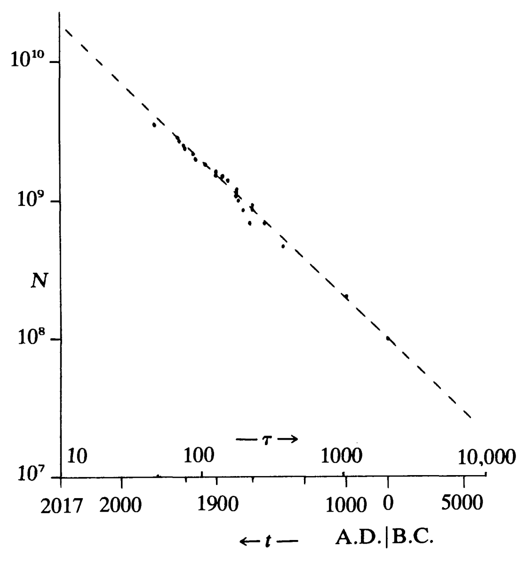


FIGURE 1.

Date, A.D.	Population in millions
0	100
1000	200
1650	545
1750	728
1800	906
1850	1171
1900	1608
1920	1834
1930	2070
1940	2295
1950	2517
1960	3005

Selected estimates of world population

TABLE 1.

In their analysis, von Foerster and his colleagues point out that one cannot discredit the population "optimists" who argue that technology has always succeeded in producing enough food to keep up with increasing population. Distribution problems aside, that is correct. "Our great-great-grandchildren will not starve to death, they will be squeezed to death." (If you are of college age now, delete one "great".) Their conclusion was that what is needed is a "population servomechanism", a feedback control device which will control P on the basis of the current value of N. That means worldwide control of population production, as yet an unsolved problem. But what is the alternative?

The Doomsday model is, of course, controversial. It has been attacked on a variety of grounds, represented by subsequent letters and articles in *Science* (**[5]**, **[10]**, **[19]**). We will examine these objections and the answers by von Foerster, Mora, and Amiot (**[5]**, **[19]**, **[28]**) in Section 6.

5. Fitting historical data by least squares

It should be clear from the previous sections that one crucial test of a conjectured model is the closeness of its fit to observable data, from which one may also infer values for the parameters in the model. If the fit is reasonable, one may then justify statements of the form "If past and present trends continue, then ...".

The least squares principle, which dates back to Gauss, may be formulated as follows. Suppose n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ (or "observations") are given to which we want to fit a functional relationship $y = f(x)$, where the form of f is known but there are one or more unknown parameters $a, b, c, \ldots$ in the description of f. It would be unrealistic to suppose that all, or perhaps any, of the data points would lie exactly on the graph of f, due to errors of observation and/or approximations in the modelling process. Thus we define "deviations"

$$d_i = y_i - f(x_i), \qquad 1 \leqq i \leqq n, \tag{13}$$

and in order to keep all of these reasonably small, we attempt to choose the parameters $a, b, c, \ldots$ so as to minimize

$$S = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} (y_i - f(x_i))^2. \tag{14}$$

The variables in equation (14) are just the parameters, since the x_i's and y_i's are known, as is the form of f. Thus we have a multivariable minimization problem, and we know from calculus that a necessary condition for minimizing S is to find "critical" values for the parameters. In short, we must solve

$$\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = \cdots = 0. \tag{15}$$

For many cases of practical interest, condition (15) turns out to be sufficient as well.

Minimizing the sum of squares of deviations (whence the name "least squares") is not the only criterion for "best fit", but there is a simple geometric reason why it is a good one. The sequence of observations $(y_1, y_2, \ldots, y_n)$ of the dependent variable may be thought of as a point in n-dimensional Euclidean space. For each choice of the parameters $a, b, c, \ldots$, the sequence $(f(x_1), f(x_2), \ldots, f(x_n))$ also represents a point in n-space. The totality of all such points constitutes a "surface" in n-space whose dimension is the number of parameters. The expression S given by (14) is the squared distance from $(y_1, y_2, \ldots, y_n)$ to a point on the surface, and the least squares principle selects a point on the model surface which is closest to the observed point.

We will illustrate the least squares process by fitting the Doomsday model (10) to the data shown in Table 1, taken from Austin and Brewer **[1]**. We should emphasize that the data shown in Table 1 are not those used by von Foerster, *et al.*, and therefore the fitted parameters need not agree with (11). The Doomsday paper **[27]** gives references to 24 data points from widely scattered sources, but the actual numbers are not given. The principle of selection in **[27]** was to reject only those estimates thought to be copied from an earlier source. Thus their data includes estimates of widely varying reliability, as well as a number of inconsistencies. Austin and Brewer, on the other hand, used only what they considered to be the most authoritative and consistent estimates, and the data in TABLE 1 are from just three different sources. This distinction will play an important role in Section 7 below, where we consider the work of Austin and Brewer in more detail. Needless to say, one would expect to obtain a better fitting curve from a given family of curves if the data is consistent than if it is not, but whether the result is more "correct" is purely conjectural.

The least squares process is easiest to carry out if the parameters appear linearly in the formal description of $y = f(x)$. We will illustrate this first with equation (12), which we write as

$$y = a + bx, \tag{16}$$

where $y = \ln N$, $a = \ln K$, $b = -k$, and $x = \ln(t_0 - t)$. If we suppose t_0 is known (an unjustified assumption, but one that is convenient for illustrating the process), then we can use $x_i = \ln(t_0 - t_i)$ and $y_i = \ln N_i$ as our data, where (t_i, N_i) are the data in TABLE 1. Substitution of (16) and (13) into (14) leads to

$$S = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2 \tag{17}$$

as the function to be minimized. S is a quadratic function of a and b with a unique minimum and no maximum, so it suffices to find the critical point as in (15). Setting the partial derivatives equal to zero, we arrive at the **normal equations** (see **[26]**, Section 14–10, Problem 1):

$$\begin{aligned} na + (\Sigma x_i)b &= \Sigma y_i, \\ (\Sigma x_i)a + (\Sigma x_i^2)b &= \Sigma x_i y_i. \end{aligned} \tag{18}$$

This illustrates the fact that if the parameters appear linearly, the optimal selection can be found by solving simultaneous linear equations.

To apply (18) to TABLE 1, we first scale the data by taking the unit of time to be 1000 years and the unit of population to be 100 million. This permits us to work with "reasonable size" numbers and avoids possible loss-of-significance problems. Using $t_0 = 2030$ (a closer guess than we would be able to make if we didn't already know the "answer"), $x_i = \ln(2.03 - t_i/1000)$, and $y_i = \ln(N_i/100)$, equations (18) become

$$\begin{aligned} 12.000\,a - 18.830\,b &= 27.517, \\ -18.830\,a + 41.745\,b &= -55.412, \end{aligned} \tag{19}$$

for which the solution is

$$a = 0.71925, \qquad b = -1.0029. \tag{20}$$

Substitution of these answers in equation (10) gives

$$N = \frac{e^{0.71925} \times 10^8}{[(2030 - t) \times 10^{-3}]^{1.0029}} = \frac{2.0529 \times 10^{.0087}}{(2030 - t)^{1.0029}} \times 10^{11} = \frac{2.0944}{(2030 - t)^{1.0029}} \times 10^{11}. \tag{21}$$

Equation (21) represents the best fit of (10) to the data of TABLE 1 *if* Doomsday is 2030 A.D.

That brings us to the sticky point of how to determine parameters that appear nonlinearly, such as t_0 in either (10) or (12). Whichever form is used, the partial derivative of S with respect to t_0 is an extremely complicated nonlinear expression. One approach to nonlinear least squares fits is to guess at starting values, perhaps by plotting the data, then expand the desired function in a Taylor series around the guessed parameters ([**26**], page 827), discard the terms of degree higher than the first, and fit to the resulting approximation, which is linear in all the parameters. The result of this fit gives new, hopefully improved, values of the parameters. The process is then repeated, each time using the new values as guesses, until convergence is obtained.

In the case of the Doomsday model, where we have a way of treating two of the three parameters linearly, it is convenient to apply the approximation process only to t_0. Thus, the computation above provides values of a and b (hence of k and K) to go with the guess $t_0^* = 2030$. Treating k and K as constants, we can expand (10) in a Taylor series in powers of $(t_0 - t_0^*)$:

$$N = K(t_0 - t)^{-k} = K[(t_0^* - t)^{-k} - k(t_0^* - t)^{-k-1}(t_0 - t_0^*) + \cdots],$$

or, as a first order approximation,

$$N = \frac{K}{(t_0^* - t)^k}\left[1 - k\frac{t_0 - t_0^*}{t_0^* - t}\right]. \tag{22}$$

(Note that N is being considered as a function of t_0, with t considered constant. Thus the differentiation is with respect to t_0, and the derivatives are evaluated at $t_0 = t_0^*$.) Since t_0 appears linearly in (22), the corresponding sum of squares S given by (17) is quadratic in t_0; setting $dS/dt_0 = 0$ leads to the following linear equation in t_0, with $\tau_i = t_0^* - t_i$:

$$\Sigma N_i \tau_i^{-k-1} - K\Sigma\tau_i^{-2k-1} + (Kk\Sigma\tau_i^{-2k-2})(t_0 - t_0^*) = 0. \tag{23}$$

We may solve equation (23) for t_0 explicitly to get our next estimate of Doomsday. That value of t_0 may then be used to recompute the x_i's in (17) and (18), leading to improved values of a and b, hence of K and k. Then we use (23) again to further refine t_0. The whole process repeats until convergence to optimal values is obtained.

TABLE 2 shows selected output from a computer run of this process, starting with a crude guess of $t_0^* = 2050$ A.D., or 2.05 in scaled value. The first column gives the iteration number. The last two columns give the root mean square error and percentage error for the fitted curve (10) with current values of the parameters. The root mean square error is the square root of the average squared deviation, or $\sqrt{S/n}$, where S is given by equation (14). This is an appropriate expression for the "average" deviation of fitted points from data points because it is minimized when S is minimized. The percentage error expresses root mean square error as a percentage of the average population in TABLE 1 (after scaling).

As in equation (21), K has to be rescaled by a factor of $10^{3(k-1)} = 1.0413$ to give

$$N = \frac{2.139}{(2030.9 - t)^{1.006}} \times 10^{11} \tag{24}$$

as the best function of the form (10) to fit the world population data in TABLE 1. If the coefficients are compared with those given by von Foerster, *et al.*, we see the effects of using different historical data: Doomsday is postponed about four years, k turns out to be slightly more than 1 instead of slightly less, and K, which represents the population one year before Doomsday, is over 21 billion rather than about 18 billion.

J	a	$k(=-b)$	$K(=e^a)$	t_0	rms	%
1	0.7358	1.079	2.087	2.0477	0.513	3.6
2	0.7335	1.070	2.082	2.0457	0.481	3.4
3	0.7315	1.063	2.078	2.0439	0.453	3.2
4	0.7299	1.057	2.075	2.0424	0.430	3.0
5	0.7284	1.051	2.072	2.0410	0.408	2.9
10	0.7240	1.030	2.063	2.0362	0.343	2.4
20	0.7208	1.013	2.056	2.0310	0.304	2.1
40	0.7199	1.007	2.054	2.0309	0.295	2.1
60	0.7198	1.006	2.054	2.0309	0.295	2.1

Selected computer output for fitting equation (10) to data in TABLE 1

TABLE 2.

6. Discussion of the Doomsday model

A number of objections have been raised to the Doomsday model by various commentators, most of which have been answered by von Foerster, Mora, and Amiot [**19**], [**5**], [**28**]. We will summarize the criticisms and responses in this section and discuss an alternate model in the next.

OBJECTION 1. A model that predicts that an obviously finite quantity will become infinite in finite time is clearly at variance with the facts and therefore irrelevant (Howland, Shinbrot [**19**]; Coale [**5**]; Austin and Brewer [**20**]).

Response. First, the model provides remarkably good fit with all available observations, the usual scientific criterion for (tentatively) accepting a hypothesis, until there is evidence for rejection. [However, we shall see that other models can fit just as well.] Second, there are many examples of accepted models of finite systems with singularities. Von Foerster and his colleagues cite the following physical models: Pressure as a function of velocity approaching the speed of sound, current approaching breakdown voltage in gaseous conduction, index of refraction in optical absorption bands, and magnetic susceptibility at Curie temperature in ferromagnetism. The usual interpretation of such a model is that one expects the system to be highly unstable in the vicinity of a singularity. It is reasonable to expect the same of a social structure subjected to extreme overpopulation.

OBJECTION 2. Not only the population but also the production rate is predicted to go to infinity on Doomsday. However, there are obvious biological factors that keep the production rate bounded. Specifically, the mortality rate cannot be reduced below zero, and the birth rate is limited by the gestation period; it could not possibly be higher than 0.75 babies per year per female of child bearing age. It is possible that biologists might learn to extend the child bearing years, shorten the gestation period, and lengthen the life span (to infinity?), but the production rate must still remain bounded. [None of the participants in the debate mentioned the possibilities of "test tube babies" or cloning, but there would still be bounds on the production rate, perhaps available glass for labware. In any case, scientists would have little incentive to drive the production rate up artificially, even if it were technically feasible.] (Robertson, Bond, and Cronkite [**19**]; Coale [**5**]; Austin and Brewer [**1**].)

Response. Different critics received different responses depending on the assumptions made about an upper bound on the production rate, but all the responses lead to very large populations in relatively short time. The boundedness of the production rate is not challenged by von Foerster, Mora, and Amiot. The most conservative bound is proposed by Coale, who notes that the world birth rate has been relatively constant at 3.9% (while the mortality rate has been declining), and in no case could the production rate conceivably exceed 6%. The Doomsday authors note that the assumption of constant birth rate of 3.9% and production rate given by (7) leads to the conclusion that the mean life span tends to infinity (i.e. the immortality problem is solved) in the year 2001. Assuming a constant 6% growth rate thereafter (i.e. reverting to the Malthusian model), one finds a world population of 30 billion in the year 2027. Knowing that it is finite is small comfort. [One might quibble about a sudden jump in birth rate from 3.9% to 6%, especially since the Doomsday model itself doesn't reach a 6% production rate until 2010, but all such computations lead to uncomfortably large population in the projected lifetimes of most who will read this.]

OBJECTION 3. There must be an upper limit on the earth's life support capabilities, and therefore the population cannot grow without bound (Hutton, Howland [**19**]; Austin and Brewer [**1**]).

Response. This is essentially the Malthusian argument pointing to impending disaster as the supportable limit is approached, a position supported by von Foerster, Mora, and Amiot. Malthus was wrong only in that he did not foresee that the maximum supportable population would be an increasing function of time, due to agricultural technology. Howland proposed that the constant λ in the Verhulst-Pearl model (4) be replaced by a linear function of t, and claimed that such a model gives a good fit to U.S. population data over the span 1790–1960. The Doomsday authors respond that no such model has been found to fit the world population over a very long period without *ad hoc* adjustments to λ from time to time. Furthermore, λ is not an observable parameter, and there is no well-developed theory of supportable population to resolve the question of the type of function λ should be. They note that one possibility for an increasing supportable population is:

(25) $$\lambda = \frac{N}{1 - \beta N^{1/k}},$$

where β is a constant. This choice has the advantages of providing a good fit over one hundred generations and of eliminating unobservable parameters from the model, since substitution of (25) into (4) leads to (8). [Of course, this evades the question of boundedness of λ.]

The question of supportable population has been dealt with in a somewhat different way by Austin and Brewer [**1**], whose model will be considered in the next section. (To our knowledge, von Foerster has not commented in print on the Austin-Brewer model.) The matter is often treated superficially. For example, Stein ([**24**] or [**25**]) suggests a limit of 40 billion, based on 10 billion acres of arable land and a requirement of one quarter acre per person for food production. He shows that this number will be reached in 2109 A.D. if the growth rate remains constant at 1.8%. (In an exercise he asks the student to show that the Malthusian model predicts a "standing room only" population in about 700 years.)

OBJECTION 4. Population forecasting by fitting mathematical curves is notably unreliable because it ignores so many important factors of demography (Dorn [**10**]). Actually, Dorn dismisses the Doomsday model in even stronger terms: "... this forecast probably will set a record, for the entire class of forecasts prepared by the use of mathematical functions, for the short length of time required to demonstrate its unreliability."

Response. Dorn's article is primarily about the "component" or "analytical" method of demography, which attempts to consider all the factors ignored in simple deterministic models, such as distributions of the population by age, sex, and geography, standards of living, and so on. The article itself shows that the analytical method has constantly led to underestimates of future population and growth rates. The Doomsday authors respond with a comparison of the projections for A.D. 2000 made by the United Nations, which Dorn considers "most authoritative", with their own projection given by (10) and (11), plus or minus 7 percent. The results are given in Table 3 (adapted from [**28**]) from which

> "... it appears that the 'most unreliable' values are just the asymptotes, at the moment of truth, to the 'most authoritative projections'; we might mention in passing that the 'most authoritative' projectors changed their minds in the last decade by roughly a factor of 2, while the 'most unreliable' values... are almost independent of the time of their derivation...".

On the matter of record short time for demonstrating the unreliability of the Doomsday model, it was observed in 1970 [**1**] and again in 1975 [**21**] that the actual world population is slightly ahead of the Doomsday projection, nearly a generation after it was made.

	From U.N. Estimate in Year				From (10) and (11)
	1950	1957	1958	1959	
Low			4.88		6.44
Medium	3.20	5.00	5.70	6.20	6.91
High			6.90	~ 7.00	7.40

Projections of population in A.D. 2000 (in billions).

TABLE 3.

OBJECTION 5. A differential equation model is inappropriate for N, since it can take only integer values. If assumption (7) about the production rate is granted, the appropriate model is a difference equation:

$$N(n) - N(n-1) = P_0 N(n-1)^{1+1/k}, \qquad n = 1, 2, 3, \ldots, \tag{26}$$

where n numbers generations. It is easy to see that equation (26) implies that N is finite for all n (Shinbrot [**19**]). (Indeed, this is clear for any equation of the form $N(n) = N(n-1) +$ any finite quantity.)

Response. It is agreed by von Foerster and colleagues that "it is unkind to perform a Dedekind cut on a man". However, equation (26) is not appropriate as a model because there are hardly any integer triples $N(n)$, $N(n-1)$, P_0 satisfying it, unless $1/k$ is an integer. "Obviously he must know such triples, and thus his suggested relationship will remain forever 'Shinbrot's last theorem'."

Shinbrot's objection to the model is easily the weakest, and it drew the weakest response, cuteness notwithstanding. In the first place, there is no reason why P_0 should be an integer, nor is there any *a priori* reason why $1/k$ should not be an integer (indeed, it is very close to 1). On the other hand, we have already observed in Section 1 why a differential equation may be an appropriate model for the growth of an integer-valued function. Human population growth is a nearly "continuous" process, not one that jumps by discrete generations. Nor is there any evidence that the production rate remains constant over a generation, as implied by (26). Thus, Shinbrot's objection betrays a fundamental lack of understanding of the modelling process.

7. The Austin-Brewer or modified coalition model

The rationale for the Doomsday model was the increasingly powerful human "coalition" against environmental forces. For this reason, the Doomsday model is also called the **coalition** model. But we noted in the previous section that both the production rate for world population and earth's capability for supporting population are bounded. Technology has been able to increase the bounds, and may continue to do so, but the bounds cannot be removed entirely. Austin and Brewer [**1**] have attempted to formulate a variant of the Doomsday or coalition model that would take these bounds into account and still fit the historical data. First they interpreted von Foerster's production rate (7) as a "fertility rate", corresponding to the factor B in equation (4). They then replaced the production rate (7) by the fertility form

$$B = A\left[1 - \exp\left(-\frac{P_0}{A} N^{1/k}\right)\right]. \tag{27}$$

By replacing the exponential function by its Maclaurin series, one can see that B is approximately $P_0 N^{1/k}$ for relatively small values of N, in accordance with (7). On the other hand, $B \to A$ as $N \to \infty$, so A represents the maximal fertility rate.

If the constant λ in the Verhulst–Pearl model (4) is interpreted as a time-independent maximum supportable population (i.e., an upper bound for supportable populations at any given time, no matter how advanced our technology becomes), then one may substitute (27) in (4) to obtain the model

$$\frac{dN}{dt} = A\left[1 - \exp\left(-\frac{P_0}{A} N^{1/k}\right)\right]\left(1 - \frac{N}{\lambda}\right) N, \tag{28}$$

which Austin and Brewer call the **modified coalition** model. Note that by wedding the "coalition" and "environmental resistance" concepts, stable equilibrium has reentered the picture. Specifically, $N = \lambda$ is an equilibrium solution of (28), and for $N < \lambda$, we have $N' > 0$, so the population increases toward equilibrium.

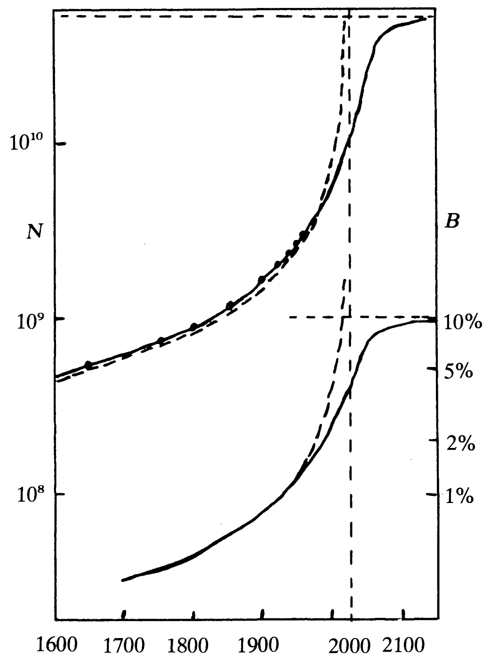
Further discussion of the properties of (28) diverges at this point from the route taken earlier. In particular, it is not possible to carry out the solution of the differential equation in closed form, and hence there is no way to apply the least squares method directly to obtain values for the parameters.

The differential equation can be solved numerically, but only when specific numerical values for the parameters are provided. Austin and Brewer followed an *ad hoc* approach of estimating the parameters, solving (28) numerically, measuring the fit with the data in TABLE 1, and then adjusting the parameters as appropriate. They reported "an excellent data fit" with the following values:

$$A = 0.1, \qquad P_0 = 5.0 \times 10^{-12}, \qquad k = 1.0, \qquad \lambda = 50 \times 10^9. \tag{29}$$

These values imply a maximal production rate of 10%, considerably larger than the 6% Coale considered the largest conceivable [**5**], and a maximum supportable population of 50 billion, considerably larger than most other estimates. Austin and Brewer note, however, that a change of 1% in k can change λ by as much as 20%.

The results of the curve fit with parameters (29) are shown graphically in FIGURE 2, adapted from [**1**]. The figure appears to show a better fit to the data points than that given by the coalition or Doomsday model. However, that is a misleading conclusion, resulting from the fact that Austin and Brewer used a different set of data. In fact, the percentage error of their fit is approximately the same as that given in Section 5 for the Doomsday model, and equation (24) would fit just as well as the solid curve in FIGURE 2.



The graphs of fertility rate *B* (lower two curves) and total population *N* (upper curves) according to the Doomsday model (dotted) and the Austin and Brewer model (solid) with data points for population taken from TABLE 1. The vertical asymptote represents "doomsday" (2027 AD) and the horizontal asymptotes represent the more hopeful stable bounds of the modified coalition model.

FIGURE 2.

Schwartz [**20**] has noted that various models can be fitted to the same data with λ ranging anywhere from 9 billion to infinity with *rms* fractional errors of no more than 2%, and therefore no conclusion can be drawn from such data that would distinguish between the coalition and modified coalition models, or that would indicate an upper bound on the supportable population. Thus we are brought back to the conclusion stated earlier, and not disputed by Austin and Brewer: Past and present trends of world population growth, no matter how they are projected into the future, point to a disastrously large population within the life span of many already born, unless a way is found to alter these trends dramatically on a worldwide scale.

We will give von Foerster, Mora, and Amiot the last words:

"... while we were displaying our wits and know-how in more or less learned discussions about the perennial question of how many angels can dance on a pin point, over ten million real people of flesh and bone, with hopes and desires, with sorrows and pain, have been added to our family of man. Our responsibility demands that we be ready with an answer when these millions ask for their right to live the span of their human condition in dignity." [**28**].

"The real problem is that today we have to prepare each single member in a family of 3 billion to face soon a decision — namely, either to persist in enjoying his children and to pay for it by having no more than two and remaining mortal, or to reach for immortality and remain childless forever. In 20 years [from 1961], of course, 4 billion will have to make this decision." [**5**].

To emphasize the conservative nature of that last statement, we add the following footnote: The Population Reference Bureau announced in March, 1976, that world population had passed the 4 billion mark.

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