15-382 Collective Intelligence – S19

Lecture 10:
Dynamical Systems 9

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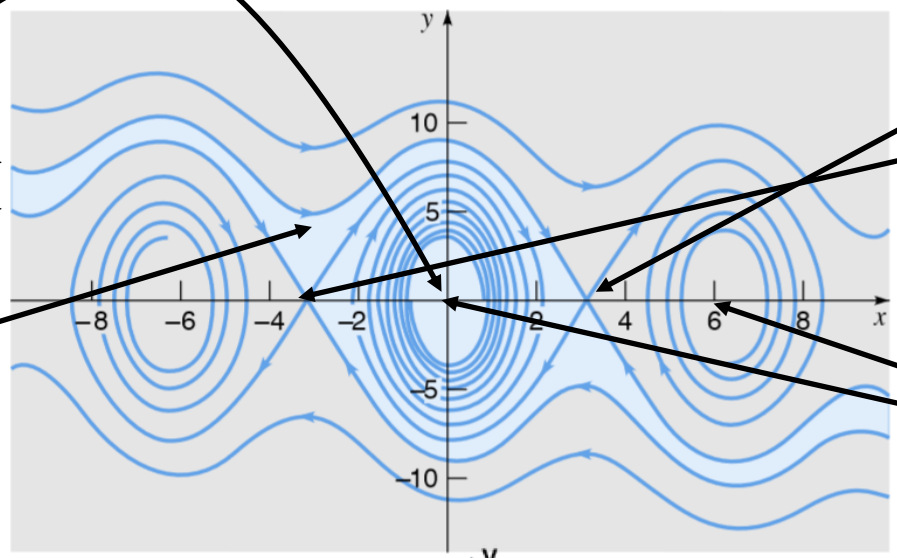
GEOMETRIES IN THE PHASE SPACE

- Damped pendulum $dx/dt = y, \qquad dy/dt = -9\sin x - \frac{1}{5}y$

One cp in the region between two separatrix

Separatrix

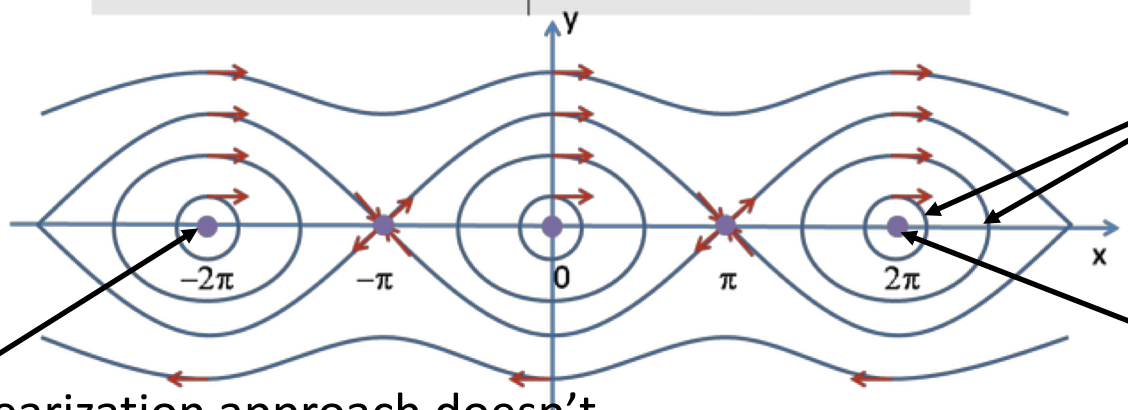
Basin of attraction



Saddle
Asymptotically unstable

Asymptotically stable spiral (or node)

- Undamped pendulum



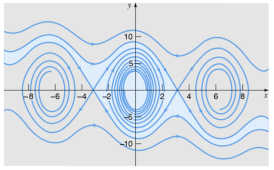
Closed orbits (periodic)

Fixed point (any period)

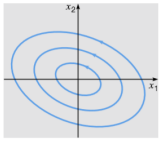
Center: the linearization approach doesn't allow to say much about stability

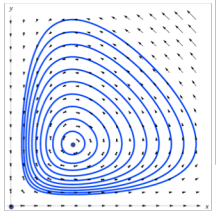
GEOMETRIES IN THE PHASE SPACE …

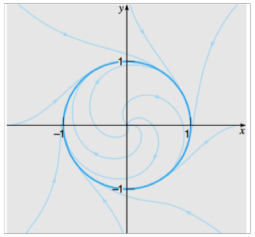
- **Question 1:** The linearization approach for studying the stability of critical points is a purely <u>local</u> approach. Going more <u>global</u>, what about the <u>basin of attraction</u> of a critical point?
- **Question 2:** When the *linearization approach fails* as a method to study the stability of a critical point, can we rely on something else?
- **Question 3:** Are critical points and *well separated closed orbits* all the *geometries* we can have in the phase space?
- **Question 4:** Does the <u>dimensionality of the phase space</u> impact on the possible geometries and limiting behavior of the orbits?
- **Question 5:** Are critical points and closed orbits the only forms of <u>*attractors*</u> in the dynamics of the phase space? Is <u>*chaos*</u> related to this?



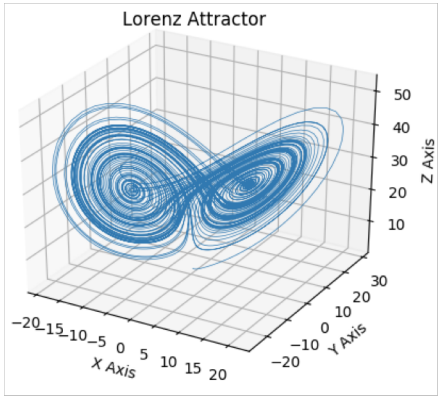
Lyapounov functions







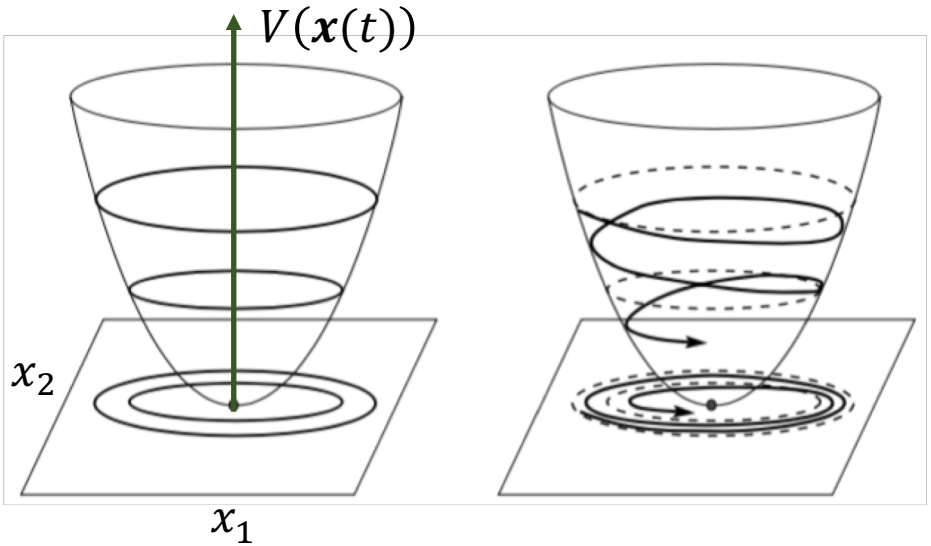
Limit cycles



LYAPUNOV DIRECT METHOD: POTENTIAL FUNCTIONS

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^n$$

$V(\boldsymbol{x}(t))$ = *Potential energy* of the system when in state $\boldsymbol{x}$, $V: \mathbb{R}^n \to \mathbb{R}$



- Time rate of change of $V(\boldsymbol{x}(t))$ along a solution trajectory $\boldsymbol{x}(t)$, we need to take the derivative of V with respect to t. Using the chain rule:

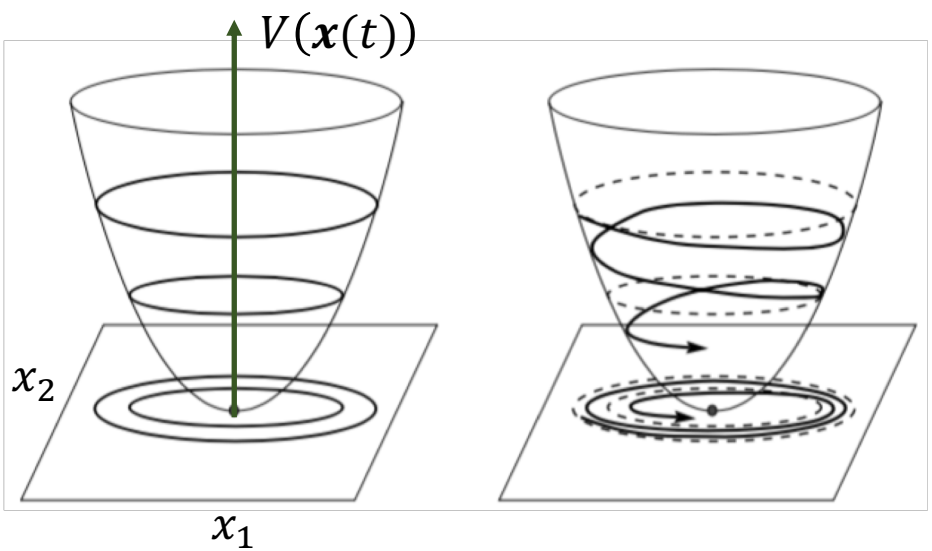
$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} = \frac{\partial V}{\partial x_1} f_1(x_1, \dots, x_n) + \cdots + \frac{\partial V}{\partial x_n} f_n(x_1, \dots, x_n)$$

Solutions do not appear, only the system itself!

LYAPUNOV FUNCTIONS

- $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^n$
- $\boldsymbol{x}^e$ equilibrium point of the system
- A function $V: \mathbb{R}^n \to \mathbb{R}$ continuously differentiable is called a **Lyapunov function** for $\boldsymbol{x}^e$ if for some neighborhood D of $\boldsymbol{x}^e$ the following hold:

  1. $V(\boldsymbol{x}^e) = 0$, and $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}^e$ in D
  2. $\dot{V}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x}$ in D
- If $\dot{V}(\boldsymbol{x}) < 0$, it's called a **strict Lyapunov function**



- $V(\boldsymbol{x}(t))$ = *Energy* of the system when in state $\boldsymbol{x}$

1. $\boldsymbol{x}^e$ is a the bottom of the graph of the Lyapunov function
2. Solutions can't move up, but can only move down the side of the potential hole or stay level

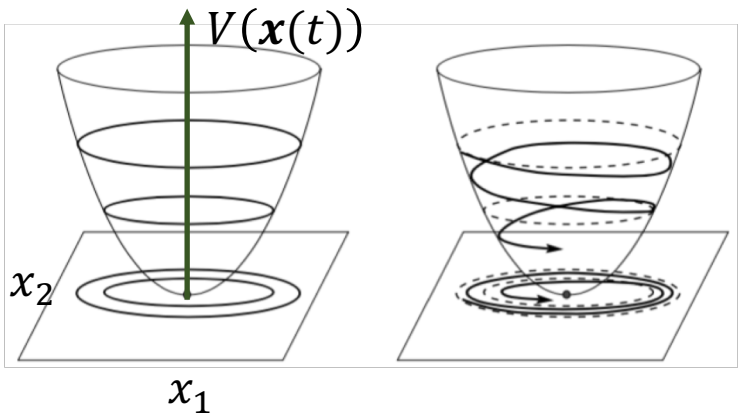
LYAPUNOV STABILITY THEOREM

- **Theorem** (Sufficient conditions for stability):

  Let $\boldsymbol{x}^e$ be an (*isolated*) equilibrium point of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

  If there exists a *Lyapunov function* for $\boldsymbol{x}^e$, then $\boldsymbol{x}^e$ is <u>stable</u>.

  If there exists a *strict Lyapunov function* for $\boldsymbol{x}^e$, then $\boldsymbol{x}^e$ is <u>asymptotically stable</u>



- Any set D on which V is a strict Lyapunov function for $\boldsymbol{x}^e$ is a subset of the basin $B(\boldsymbol{x}^e)$
- If there exists a strict Lyapunov function, then there are no <u>closed orbits</u> in the region of its basin of attraction

- *Definition:* Let $\boldsymbol{x}^e$ be an asymptotically stable equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Then the **basin of attraction** of $\boldsymbol{x}^e$, denoted $B(\boldsymbol{x}^e)$, is the set of initial conditions $\boldsymbol{x}_0$ such that the associated orbits asymptotically converge to $\boldsymbol{x}^e$:
  $\lim_{t\to\infty} \boldsymbol{F}(\boldsymbol{x}_0, t) = \boldsymbol{x}^e$

How do we define Lyapunov functions?

- **Physical systems:** Use the <u>energy function of the system itself</u>

For a damped pendulum ($x = \theta, y = \frac{d\theta}{dt}$)

$$V(x, y) = mgL(1 - \cos x) + \tfrac{1}{2}mL^2y^2.$$

- **Other systems:** Guess!

$$dx/dt = -x - xy^2$$

$$dy/dt = -y - x^2y$$

$$\boldsymbol{x}^e = (0,0)$$

$$V(x, y) = ax^2 + bxy + cy^2$$

The function

$$V(x, y) = ax^2 + bxy + cy^2$$

is positive definite if, and only if,

$$a > 0 \quad \text{and} \quad 4ac - b^2 > 0,$$

and is negative definite if, and only if,

$$a < 0 \quad \text{and} \quad 4ac - b^2 > 0.$$

$$V_y(x, y) = bx + 2cy.$$

$$V_x(x, y) = 2ax + by.$$

$$\begin{aligned}\dot{V}(x, y) &= (2ax + by)(-x - xy^2) + (bx + 2cy)(-y - x^2y) \\ &= -[2a(x^2 + x^2y^2) + b(2xy + xy^3 + x^3y) + 2c(y^2 + x^2y^2)]\end{aligned}$$

For $b = 0, \quad a, c > 0 \quad \rightarrow \dot{V} < 0, \quad V > 0 \qquad \Rightarrow (0,0)$ is asymptotically stable