16-311-Q Introduction to Robotics

Lecture 10A: Feedback-based Control 2A

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TO BE DONE ...

Stability properties of linear systems
Linearization of previous control systems
Stability domain for feedback-based gains
Other types of controllers?

CONTROLLABILITY OF A DYNAMICAL SYSTEM

Time-invariant dynamical system with *m control inputs* $\mathbf{u}$

$$\dot{\mathbf{x}} = f\left(\mathbf{x}(t), \mathbf{u}(t)\right), \quad \mathbf{x} \in \mathbb{R}^n, \ \mathbf{u} \in \mathbb{R}^m$$

Control inputs are defined according to a **feedback law**: $\boldsymbol{u}(t) = K\boldsymbol{x}(t), \quad K$ is an $n \times m$ feedback Gain matrix

Controllability: Any initial state $\boldsymbol{x}(0)$ can be steered to any final state $\boldsymbol{x}^1$ at a finite time t_1 based on the inputs from the feedback law.

For a robot: All configurations can be achieved in finite time from a given initial configuration.

Note: *The trajectory between 0 and t_1 is not specified*

For linear dynamical systems

$$f\left(\mathbf{x}(t), \mathbf{u}(t)\right) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

algebraic criteria for controllability are available:

$C = [\ B \ AB \ A^2B \ \cdots \ A^{n-1}B\ ], \quad \mathrm{rank}(C) = n$ (C has full rank)

For non-linear dynamical systems, general controllability criteria are not available!

Local (in space and time) notions of controllability are employed

STABILITY OF A DYNAMICAL SYSTEM

Equilibrium: A state $\boldsymbol{x}^e$ of is said to be an *equilibrium state* if and only if

$$\boldsymbol{x}^e = \boldsymbol{x}(t;\, \boldsymbol{x}^e,\, u(t)=0) \text{ for all } t \geq 0.$$

If a trajectory reaches an equilibrium state and if no input is applied the trajectory will stay at the equilibrium state forever (internal system's dynamics doesn't move the system away from the equilibrium point)

For a linear system the zero state is a always an equilibrium state

Stable equilibrium: An equilibrium state $\boldsymbol{x}^e$ is said to be *stable* if and only if for any positive ε, there exists a positive number δ(ε) such that the inequality

$$\|\boldsymbol{x}(0) - \boldsymbol{x}^e\| \leq \delta$$

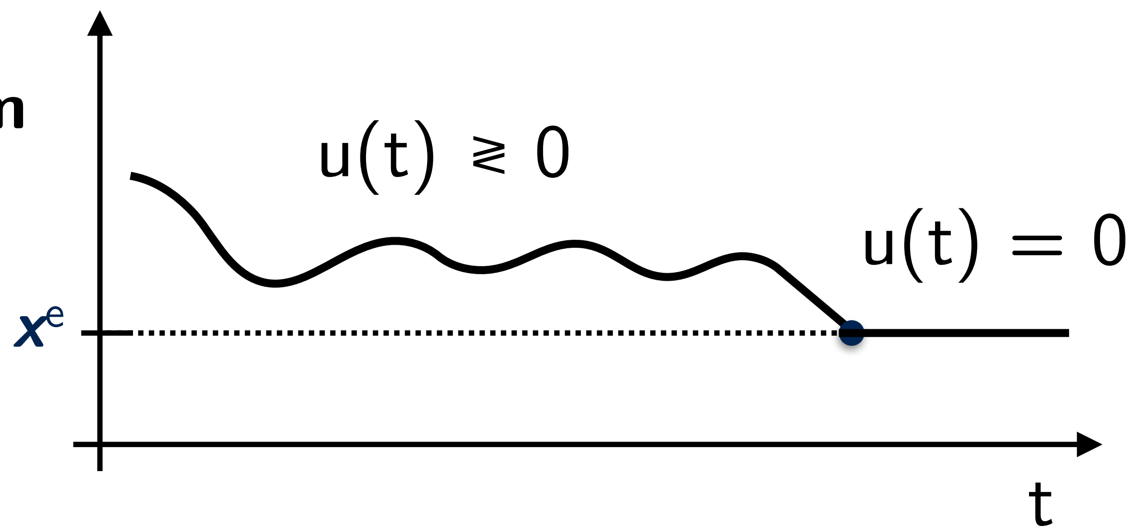
Lyapunov stability

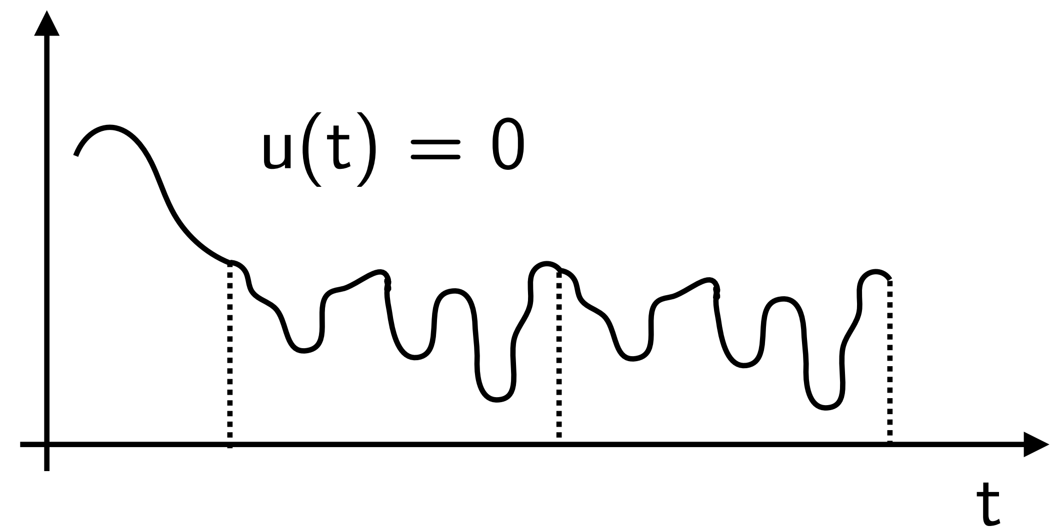
implies that $\|\boldsymbol{x}(t;\, \boldsymbol{x}(0),\, u(t)=0) - \boldsymbol{x}^e\| \leq \varepsilon$ for all $t \geq 0$.

An equilibrium state $\boldsymbol{x}^e$ is stable if the response following after starting at any initial state x(0) that is sufficiently near to $\boldsymbol{x}^e$ will not move the state far away from $\boldsymbol{x}^e$.

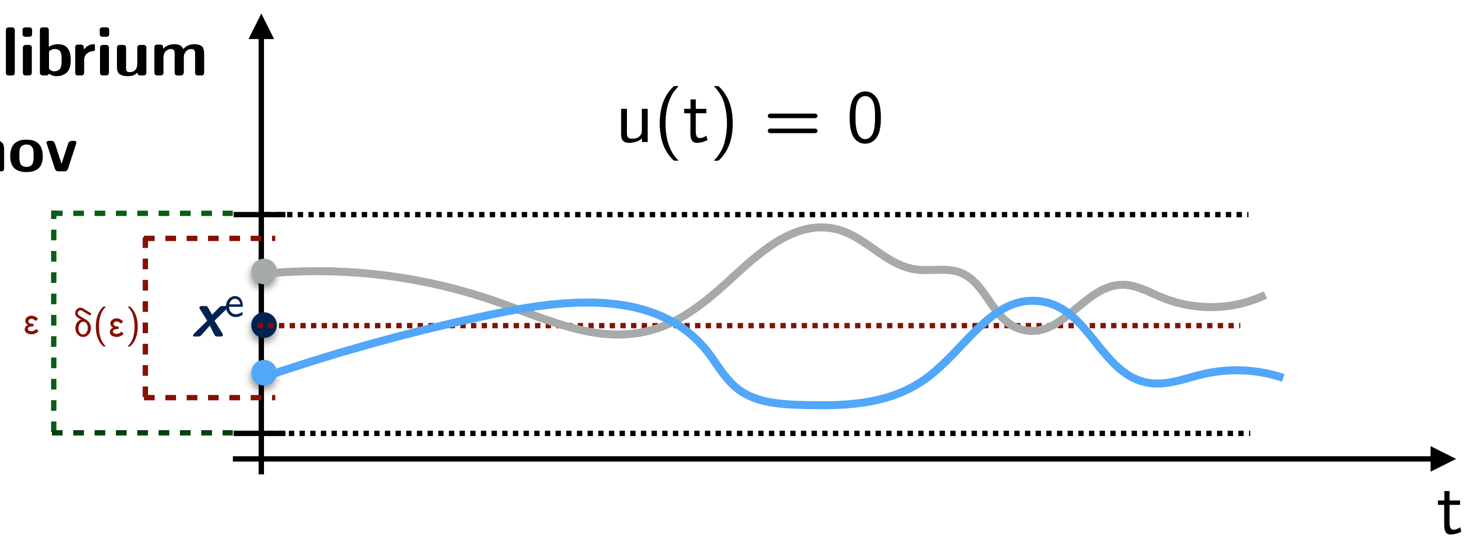
Asymptotically stable equilibrium: If the equilibrium $\boldsymbol{x}^e$ is Lyapunov-stable and if every motion starting sufficiently near to $\boldsymbol{x}^e$ converges (go back) to $\boldsymbol{x}^e$ as $t \to \infty$.

Equilibrium

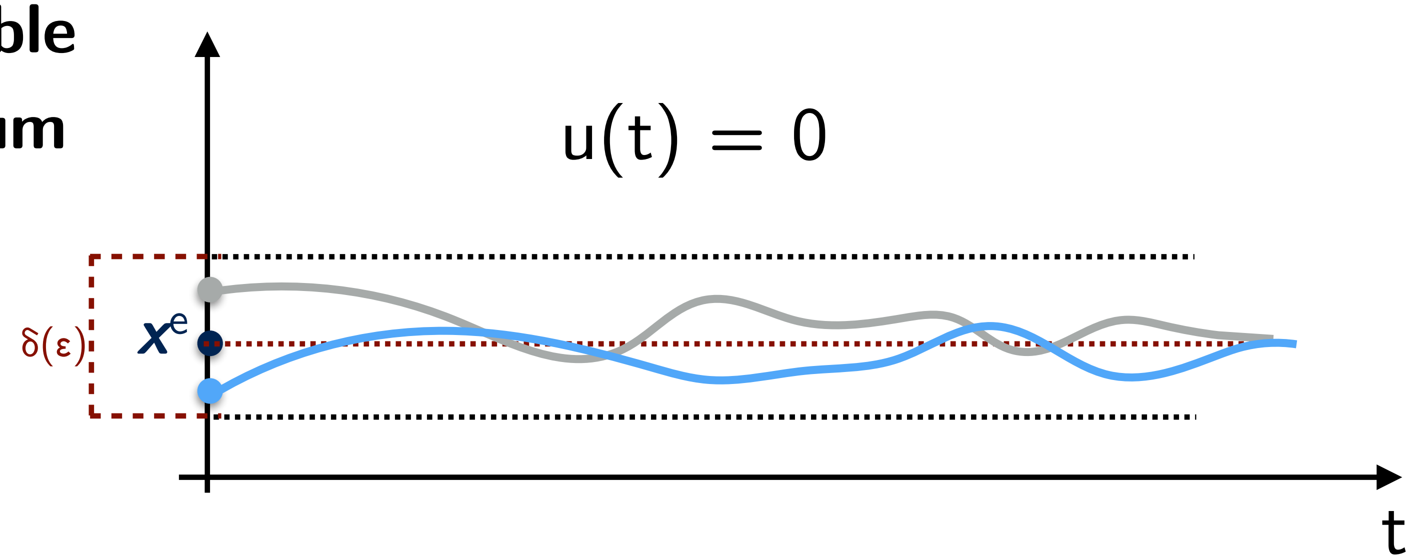




Stable equilibrium
Lyapunov



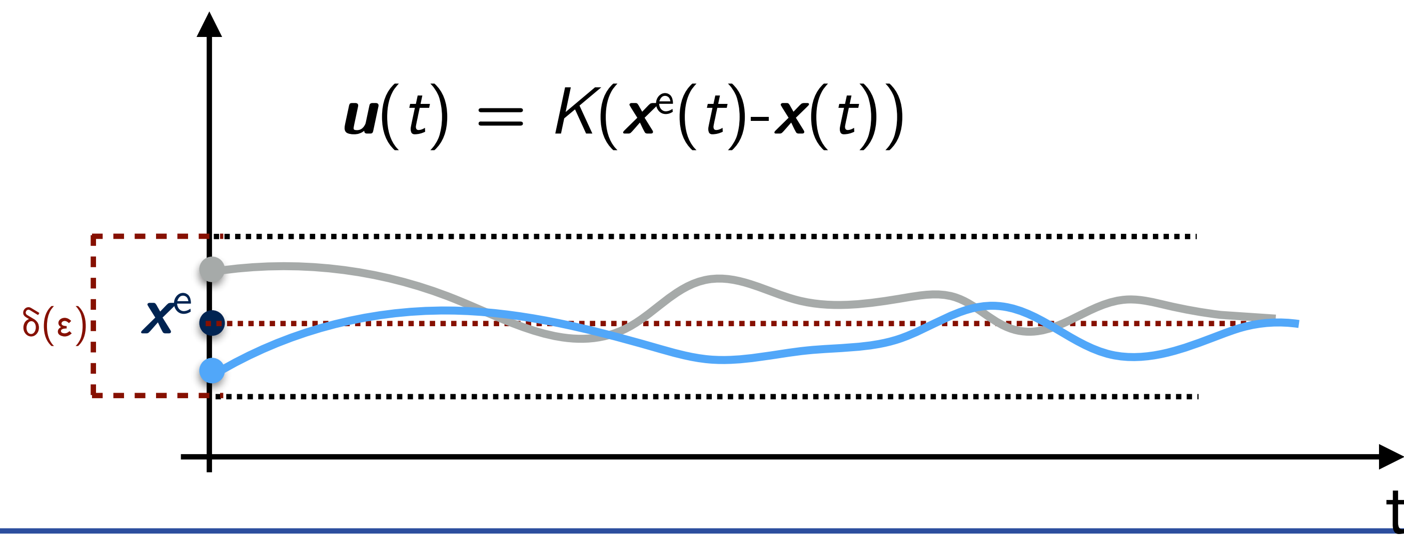
Asymptotically Stable
Lyapunov equilibrium



CONTROLLABILITY VS. STABILIZABILITY

Stabilizability: The problem of finding a *feedback control law* so as to make a closed-loop equilibrium point $\boldsymbol{x}^e$ or admissible trajectory $\boldsymbol{x}^e(t)$ *asymptotically stable*.

Stabilizability is very important in practice to be able to cope with real-world disturbances



Stabilizability of Linear dynamic systems:

Controllability implies asymptotic (actually, exponential) stabilizability by a smooth state feedback law. In fact, the controllability condition implies that there exist choices of the constant gain matrix K such that the linear P control

$$\boldsymbol{u}(t) = K(\boldsymbol{x}^e(t)\text{-}\boldsymbol{x}(t))$$

makes $\boldsymbol{x}^e(t)$ asymptotically stable

Linear systems:
Controllability ➔ Stabilizability

Non-Linear systems:
Controllability **?** Stabilizability

LINEAR DYNAMIC SYSTEMS

Given a linear dynamic system in the form of a homogeneous system of ODEs:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$$

- Where will the system state go? ➔ Solve the system to find the time-dependent function $\boldsymbol{x}(t)$ for the evolution of the state variables.

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The solution function for the state variables is the following, with c_1 and c_2 integration constants depending on the initial point $\boldsymbol{x}(0)$:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{12t} + c_2 4e^{-6t} \\ c_1 e^{12t} - c_2 5e^{-6t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{12t} + c_2 \begin{bmatrix} 4 \\ -5 \end{bmatrix} e^{-6t}$$

A's **eigenvalues** $\boxed{\lambda_1 = 12, \ \lambda_2 = -6}$

$$\boxed{r_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, r_2 = \begin{bmatrix} 4 \\ -5 \end{bmatrix}}$$ *A*'s **eigenvectors**

In the general case of a system with n state variables (equations)

$$\boxed{\mathbf{x}(t) = c_1\mathbf{r}_1 e^{\lambda_1 t} + c_2\mathbf{r}_2 e^{\lambda_2 t} + \ldots c_n\mathbf{r}_n e^{\lambda_n t}}$$

- Where are the equilibrium points? Answering means to first find the **fixed points** (the **attractors**, in general) that correspond to setting to zero the rate of variability of the state variables. In the example, the fixed points are the solutions of:

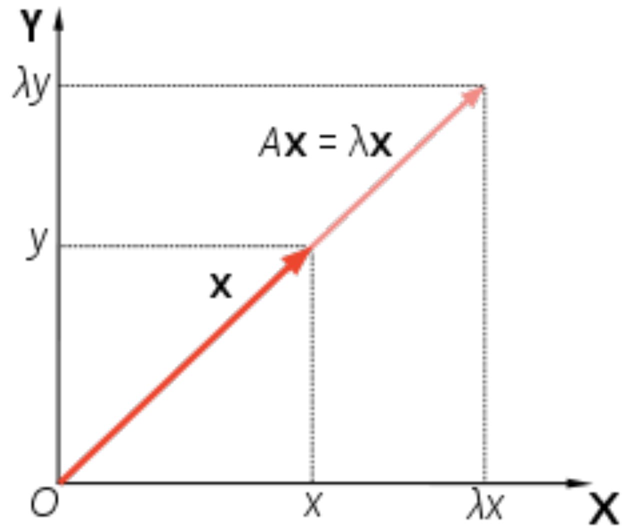
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

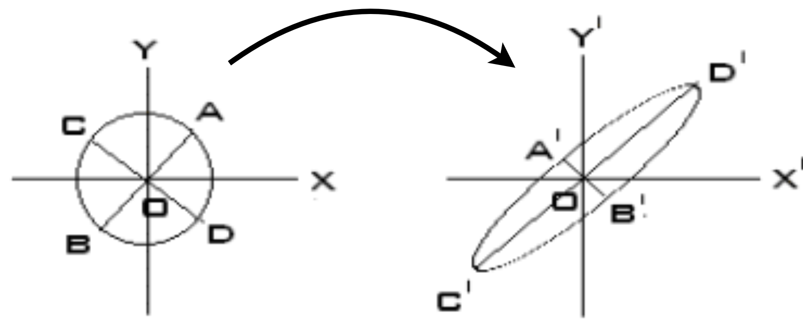
- What will the system do in correspondence of the fixed points? Is the system (asymptotically) *stable*? If we place the system close to a fixed point, or, similarly, we disturb a system at a fixed point, will the system go back to the fixed point, or will it diverge from it? What about the behavior at any other point?

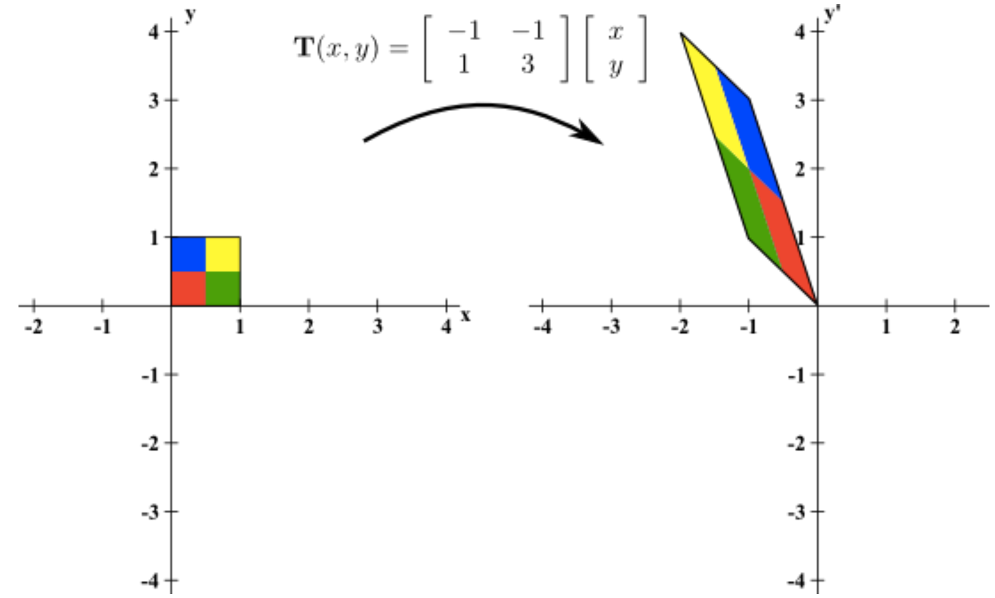
For a **linear system**, a stability analysis for the fixed points can be performed through the calculation of the eigenvalues of the matrix of the coefficients.

The eigenvalues are the solution of the **characteristic equation $\det(A - \lambda I) = 0$,** and determine the time evolution of the system along the principal directions of the eigenvectors

RECAP ON EIGENVALUES AND EIGENVECTORS







Symmetric matrix:
eigenvectors forms an orthogonal basis

Elementary Matrices for Linear Transformations in R^2

Reflection in y-Axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in x-Axis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Reflection in Line $y = x$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Horizontal Expansion ($k > 1$) or Contraction ($0 < k < 1$)

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical Expansion ($k > 1$) or Contraction ($0 < k < 1$)

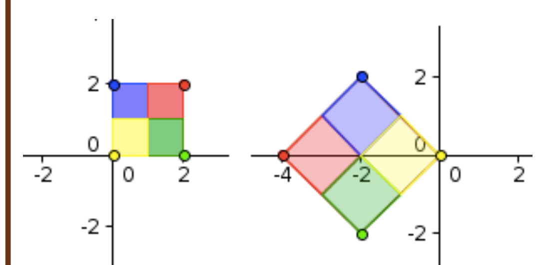
$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

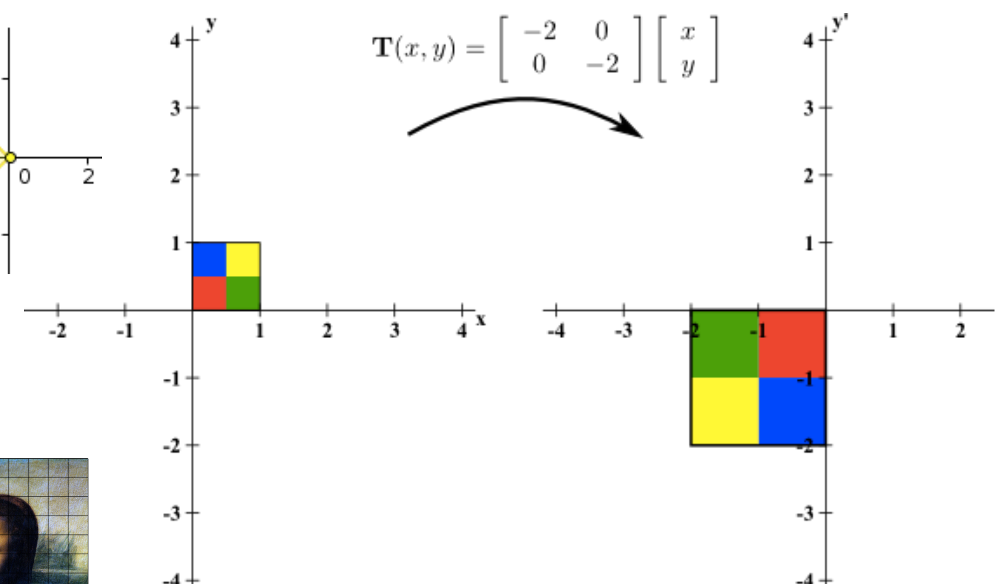
Horizontal Shear

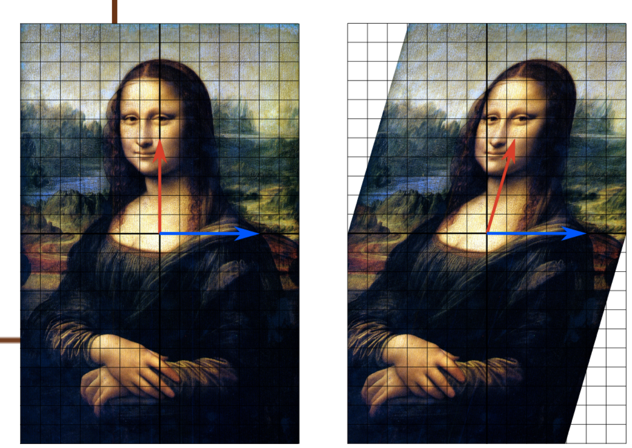
$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

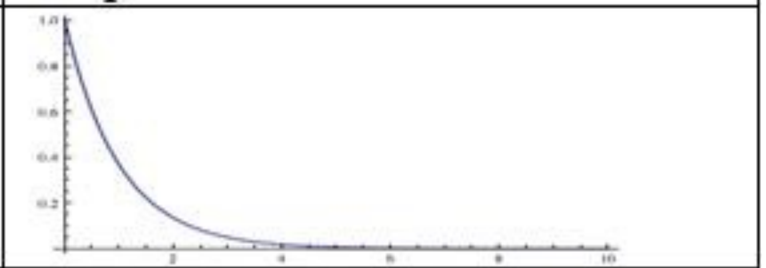
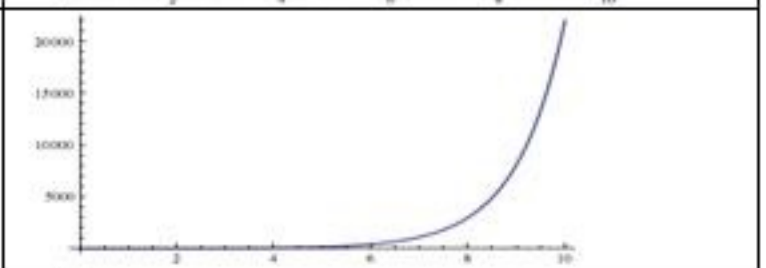
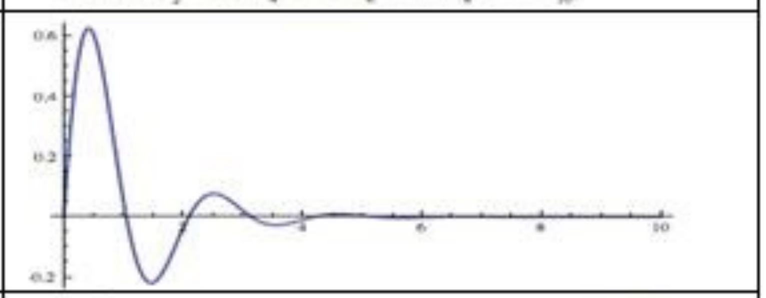
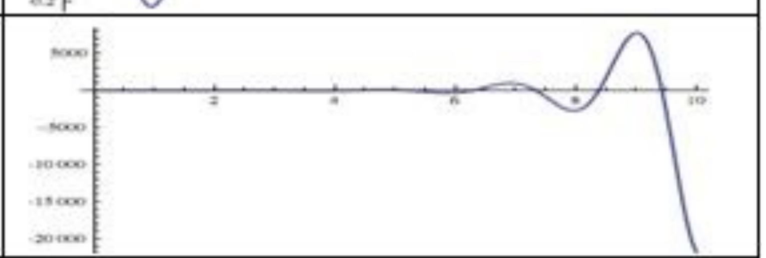
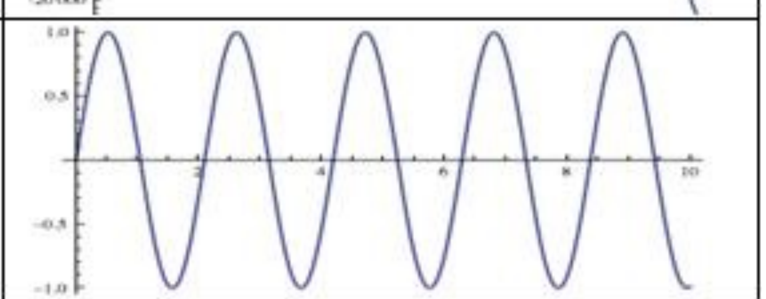




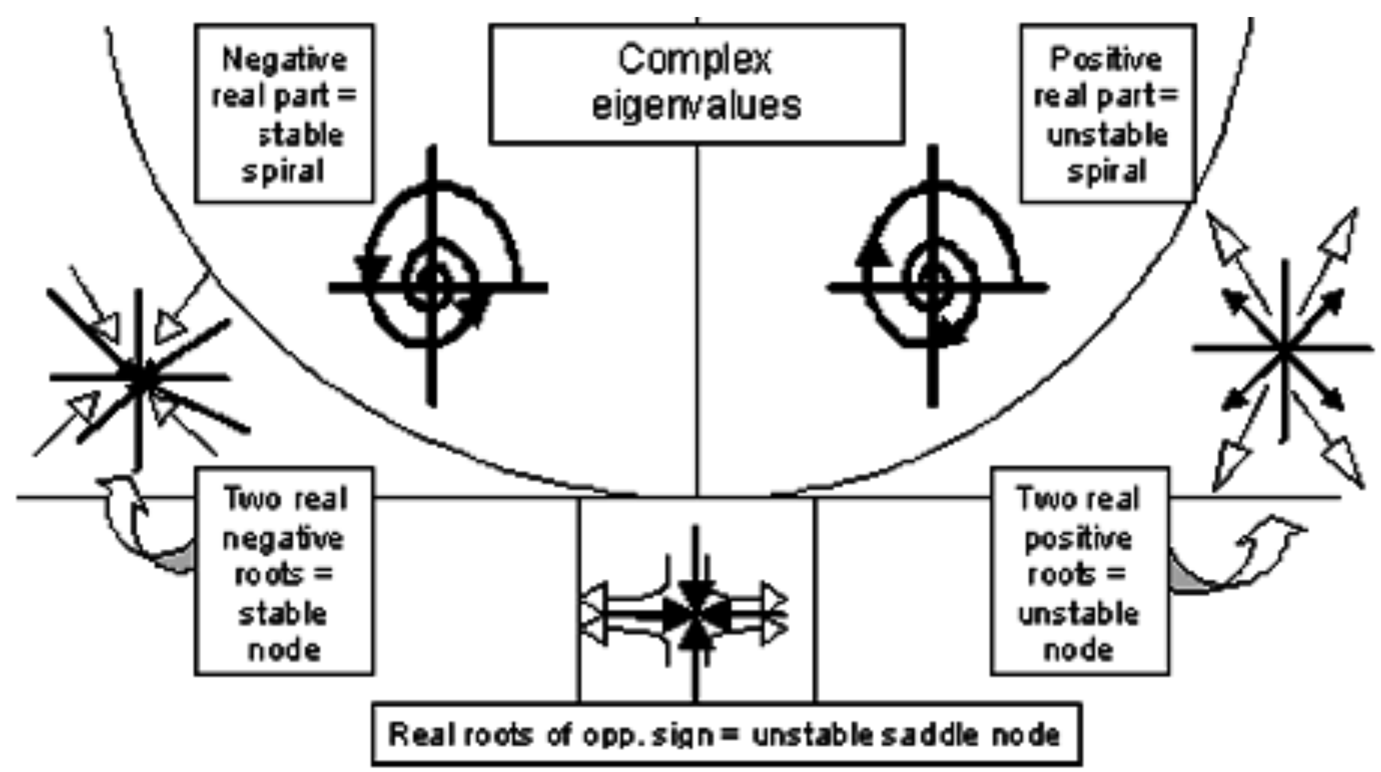


(LINEAR) STABILITY BEHAVIOR VS. EIGENVALUES

State behavior in the vicinity of a fixed point in relation to its stability based on the eigenvalues (the value of y axis quantifies the distance from the fixed point)

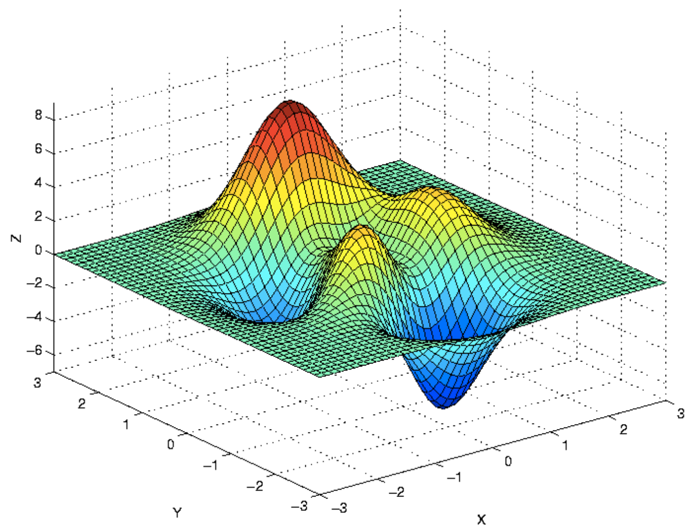
Eigenvalues	Graph
All real and negative	
All real and one or more are positive	
All real eigenvalues are negative and there are imaginary parts	
One or more eigenvlaues have a positive real part and there are imaginary parts	
Real parts of the eigenvlaues are zero and there are imaginary parts	

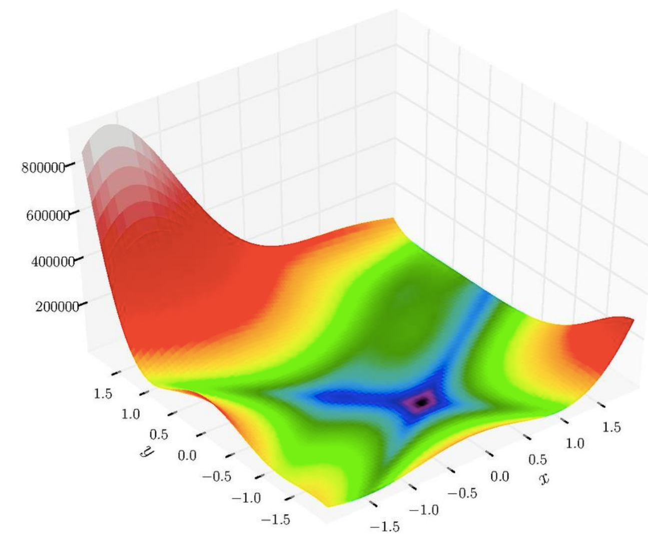
Eigenvalue Type	**Stability**	**Oscillatory Behavior**	**Notation**
All Real and +	Unstable	None	Unstable Node
All Real and -	Stable	None	Stable Node
Mixed + & - Real	Unstable	None	Unstable saddle point
+a + b*i*	Unstable	Undamped	Unstable spiral
-a + b*i*	Stable	Damped	Stable spriral
0 + b*i*	Unstable	Undamped	Circle
Repeated values	Depends on orthogonality of eigenvectors		

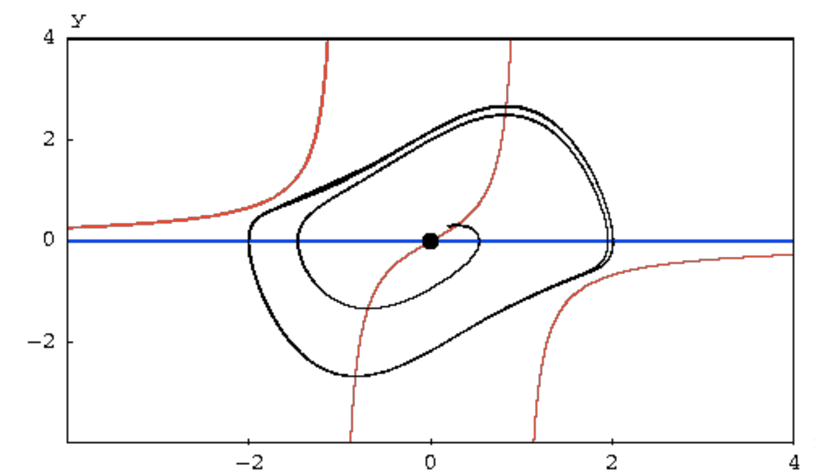


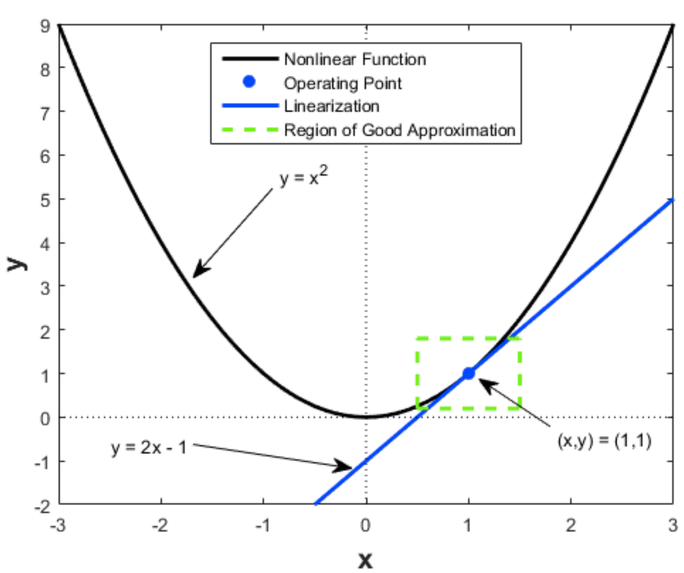
NON-LINEAR DYNAMICAL SYSTEMS?

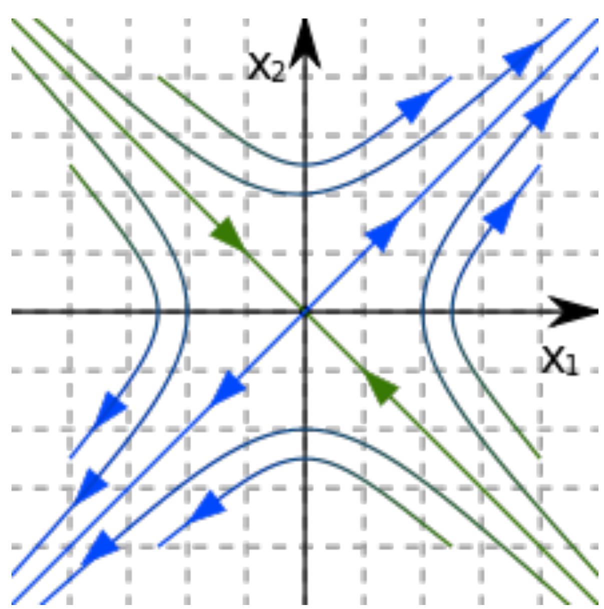
A ***non-linear system can be linearized around a (fixed) point***, and studied with the same methods. The effectiveness of the linearization decreases with the distance from the fixed point itself and with the stability characteristics of the point .











Hyperbolic point (near the point trajectories resemble hyperbolas) Saddle point

Hartman-Grobman theorem: In a hyperbolic equilibrium point where all eigenvalues have non-zero real parts, the flows of the linearized and nonlinear system are (topologically) equivalent near the equilibrium. In particular, the stability of the nonlinear equilibrium is the same as the stability of the equilibrium of the linearized system.

.... OUR LINEAR FEEDBACK CONTROL LAWS?

This is what we have derived:

$$v(t) = K_\rho \rho(t)$$

$$\gamma(t) = K_\alpha \alpha(t) + K_\beta \beta(t)$$

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos(\alpha) & 0 \\ \frac{\sin(\alpha)}{\rho} & -1 \\ -\frac{\sin(\alpha)}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -K_\rho \rho \cos(\alpha) \\ -K_\rho \sin(\alpha) - K_\alpha \alpha - K_\beta \beta \\ -K_\rho \sin(\alpha) \end{bmatrix}$$

?? Asymptotically stable as long as:

$$K_\rho > 0, \quad K_\beta < 0, \quad K_\alpha - K_\rho > 0$$

It's NOT linear in the state $[\rho(t)\ \alpha(t)\ \beta(t)]$

Linearization

around the fixed point [0 0 0]

LINEARIZATION OF THE CONTROL LAW

$$\begin{bmatrix} \dot\rho \\ \dot\alpha \\ \dot\beta \end{bmatrix} = \begin{bmatrix} -K_\rho \rho \cos(\alpha) \\ -K_\rho \sin(\alpha) - K_\alpha \alpha - K_\beta \beta \\ -K_\rho \sin(\alpha) \end{bmatrix}$$

In a small neighborhood of [0 0 0]:
cos(x) = 1, sin(x) = x

$$\begin{bmatrix} \dot\rho \\ \dot\alpha \\ \dot\beta \end{bmatrix} = \underbrace{\begin{bmatrix} -K_\rho & 0 & 0 \\ 0 & -(K_\alpha - K_\rho) & -K_\beta \\ 0 & -K_\rho & 0 \end{bmatrix}}_{A} \begin{bmatrix} \rho \\ \alpha \\ \beta \end{bmatrix}$$

Linearization
around the fixed point [0 0 0]

The characteristic polynomial of the coefficient (gain) matrix A is

$$(\lambda + K_\rho)(\lambda^2 + \lambda(K_\alpha - K_\rho) - K_\rho K_\beta)$$

all roots have negative real part (i.e., stability) if

$$K_\rho > 0, \quad K_\beta < 0, \quad K_\alpha - K_\rho > 0$$

For ***robust pose control***, the following strong stability conditions ensures that the robot *does not change direction approaching the goal*, implying that conditions on α

$$K_\rho > 0, \quad K_\beta < 0, \quad K_\alpha + \frac{5}{3}K_\beta - \frac{2}{\pi}K_\rho > 0$$

If $\alpha(0) \in \ I_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \ \alpha(t) \in \ I_f \ \forall t$

If $\alpha(0) \in \ I_b = \bar{I}_f \Rightarrow \ \alpha(t) \in \ I_b \ \forall t$

FUNCTION LINEARIZATION

- Linearization requires first-order differenziability. For a scalar function $f(x)$ of one variable, the Taylor series of the 1st order in a point x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)$$

  is the same as the equation of the tangent line in x_0 (up to an error of the second order), that is, the derivative defines the linear approximation of the function.

- For a scalar function of multiple variables, the gradient vector $\nabla f(x)$ generalizes the notion of derivatives for all variables, and linearization is performed using the same Taylor's series but using the gradient instead of the derivative.

- For a vector function $g(x) : \mathbb{R}^n \mapsto \mathbb{R}^p$, the relation still holds, but the gradient vector is substituted by the Jacobian matrix:

$$\boldsymbol{J}(\boldsymbol{x}) = [\nabla g_1(\boldsymbol{x}) \;\; \dots \;\; \nabla g_p(\boldsymbol{x})]^T = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \dots & \frac{\partial g_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial g_p}{\partial x_1} & \frac{\partial g_p}{\partial x_2} & \dots & \frac{\partial g_p}{\partial x_n} \end{pmatrix}, \quad \boldsymbol{J}^T(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \dots & \frac{\partial g_p}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \dots & \frac{\partial g_p}{\partial x_2} \\ \dots & \dots & \dots & \dots \\ \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \dots & \frac{\partial g_p}{\partial x_n} \end{pmatrix}$$

  - A dynamic system of p equations in n state variables can be precisely seen as a vector function, with the Jacobian showing how every variable changes with the variation of another variable derivatives
  - The general approach for the linearization of a dynamic system passes through the writing of the Jacobian matrix at the fixed point of interest x_0, and use of the resulting Jacobian to write the Taylor's series of the first order that linearly approximates the system in x_0