16-311-Q Introduction to Robotics

Lecture 7: DOFs vs. Maneuverability Kinematics Equations

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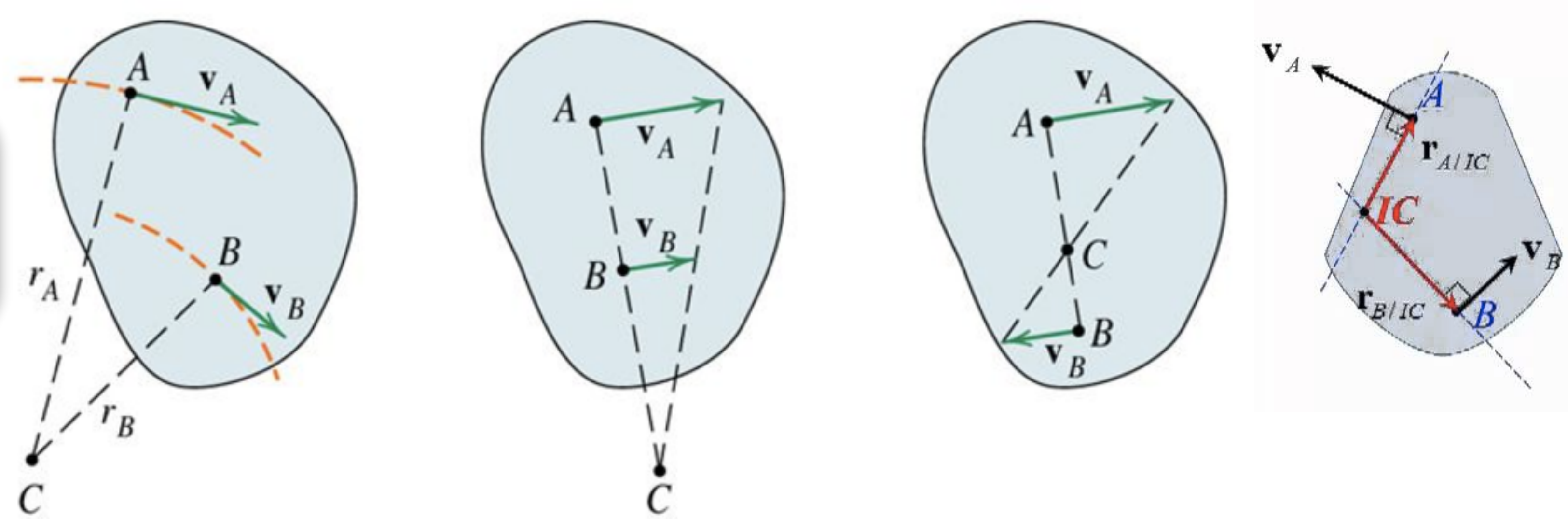
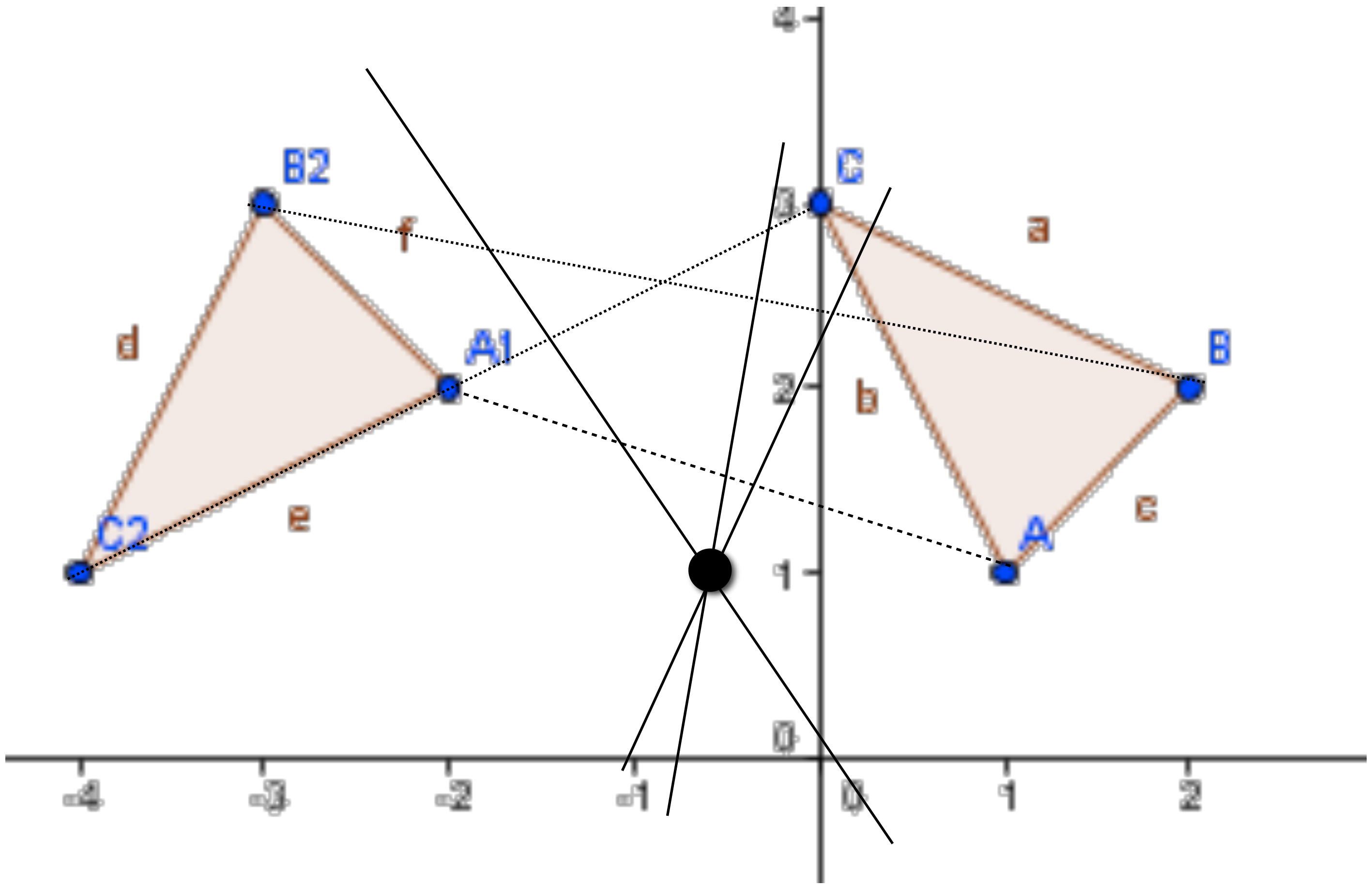
INSTANTANEOUS CENTER OF ROTATION

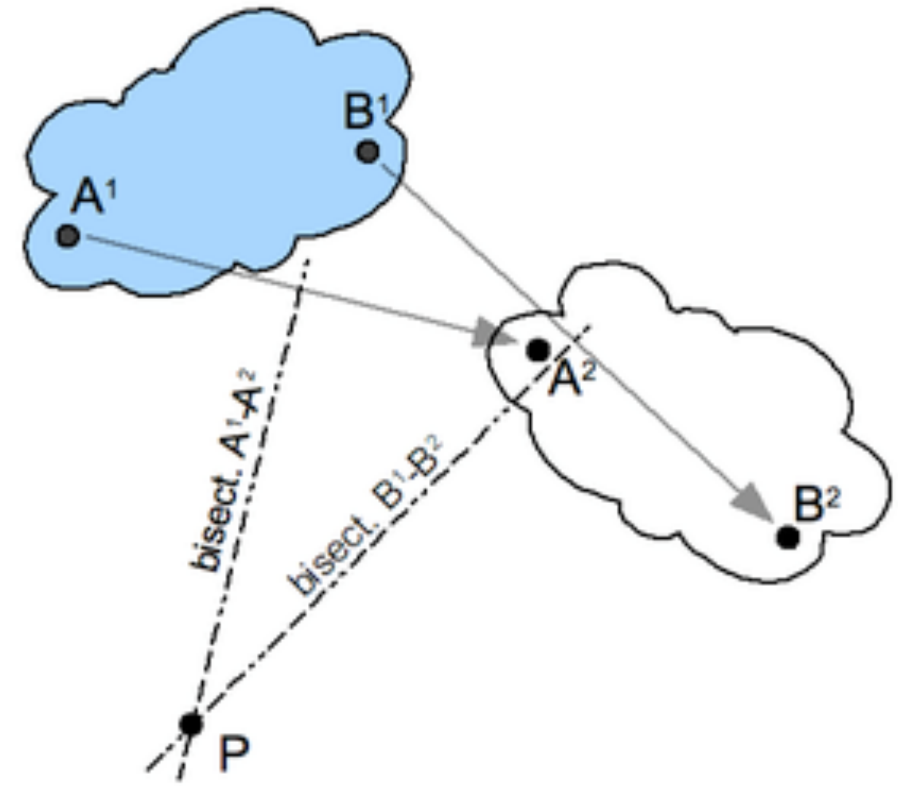
$$
{}^{R}\boldsymbol{r}_{ICR} = \frac{1}{\omega^2}(\boldsymbol{\omega} \times {}^{W}\boldsymbol{v}_R)
$$

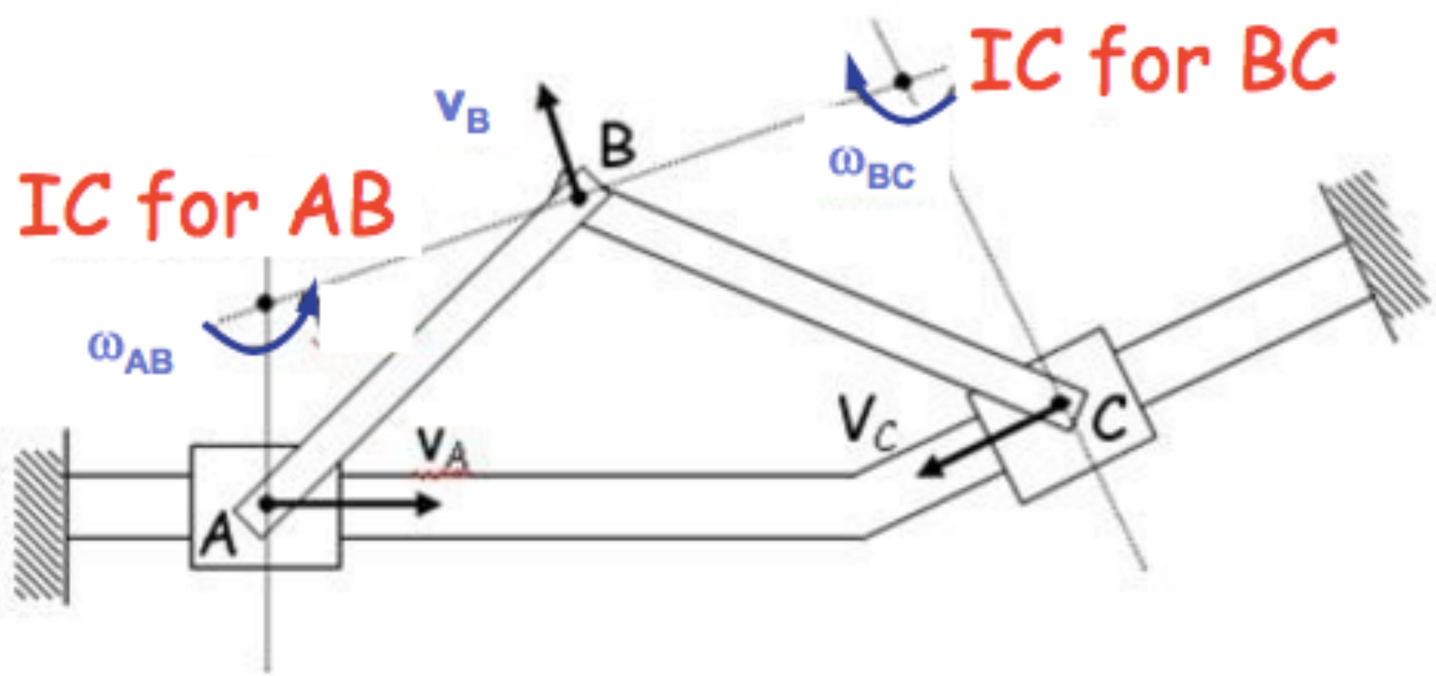


- If the there is no translation, the ICR is the same as the center of rotation: the velocity of {R} is zero in {W} and accordingly, the ICR coincides with {R}.
- The position vector of the ICR is perpendicular to $\mathbf{v}_\mathbf{R}$, the velocity vector in {R}. More in general, selected a point **A** in the body, the position vector (**ICR — A**) is perpendicular to the velocity vector in **A**
- $\rightarrow$ If we know the velocity at two points of the body, **A** and **B**, then the location of **ICR** can be determined geometrically as the intersection of the lines which go through points **A** and **B** and are perpendicular to $\mathbf{v}_\mathbf{A}$ and $\mathbf{v}_\mathbf{B}$
- When the angular velocity, $\boldsymbol{\omega}$, is very small, the center of rotation is very far away; when it is zero (i.e. a pure translation), the center of rotation is at infinity.

GEOMETRIC CONSTRUCTION, MULTI-PARTS BODIES





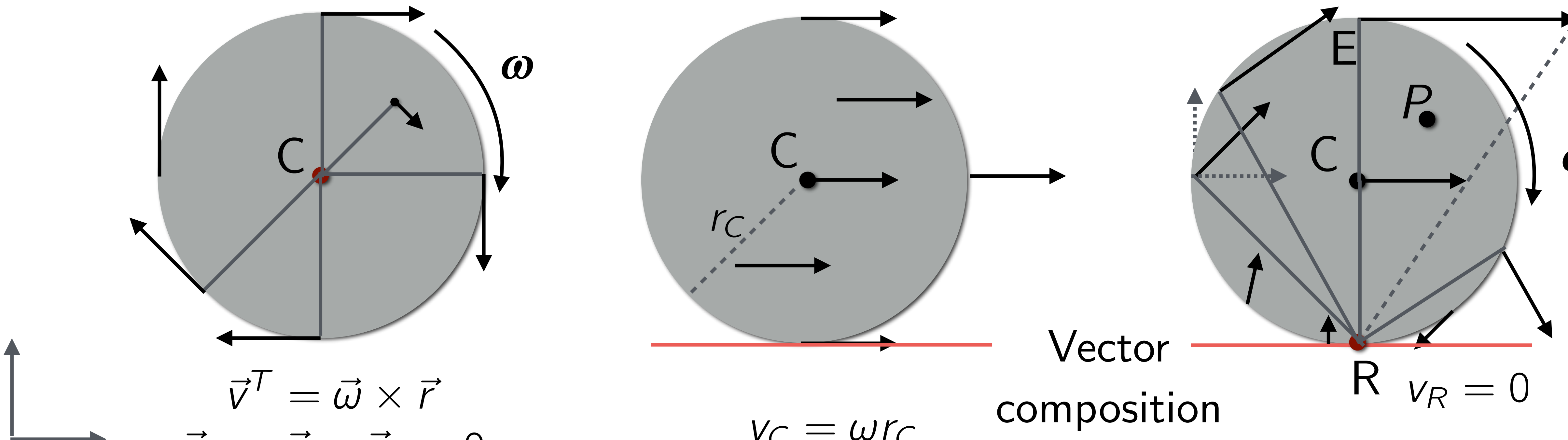


ICR FOR WHEELS

Pure rotation

Pure translation

Pure rolling:
rotation + translation



$\vec{v}^T = \vec{\omega} \times \vec{r}$

$\vec{v}_C = \vec{\omega} \times \vec{r}_C = 0$

$v_C = \omega r_C$

Vector composition at each point

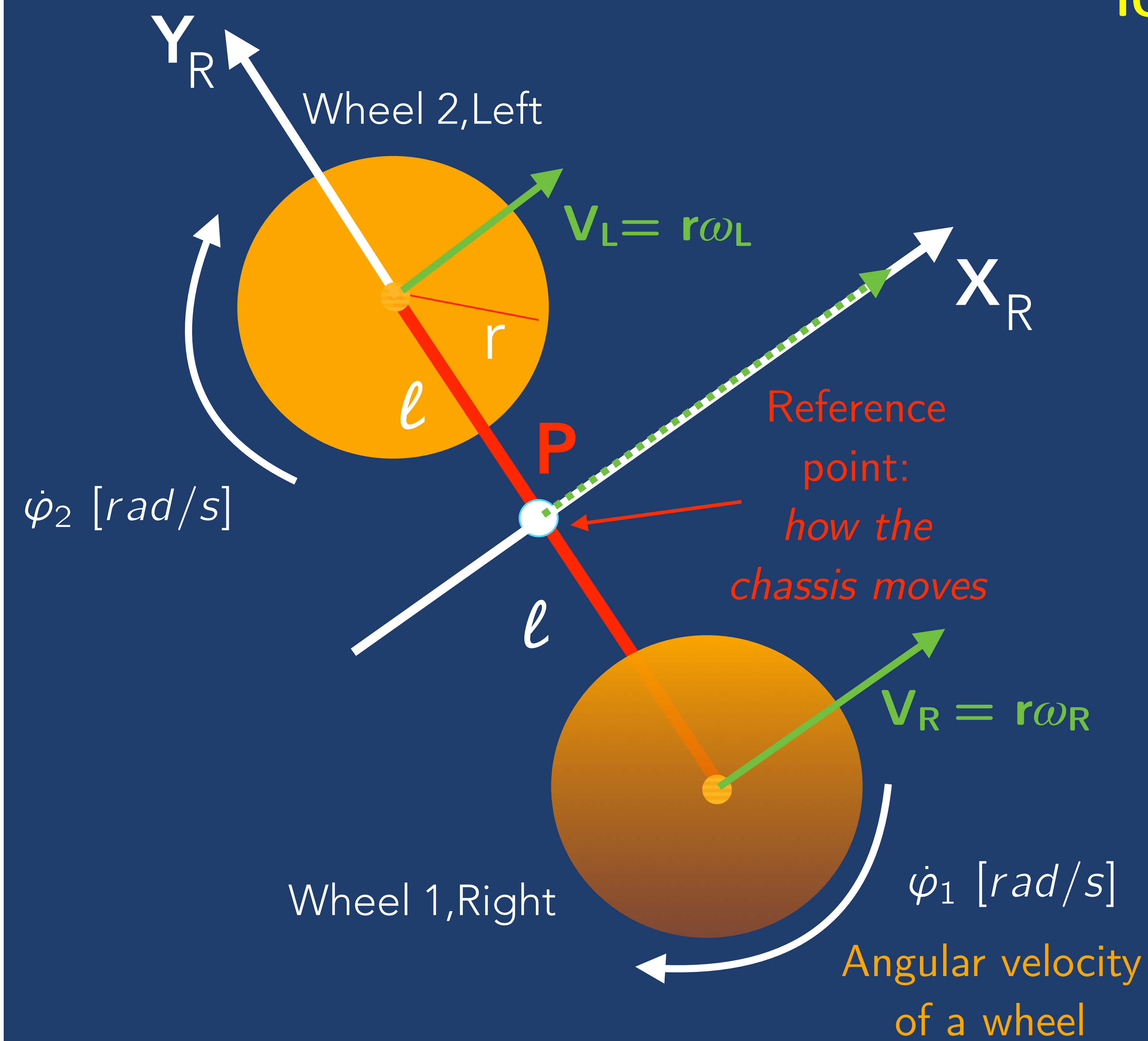
$v_R = 0$

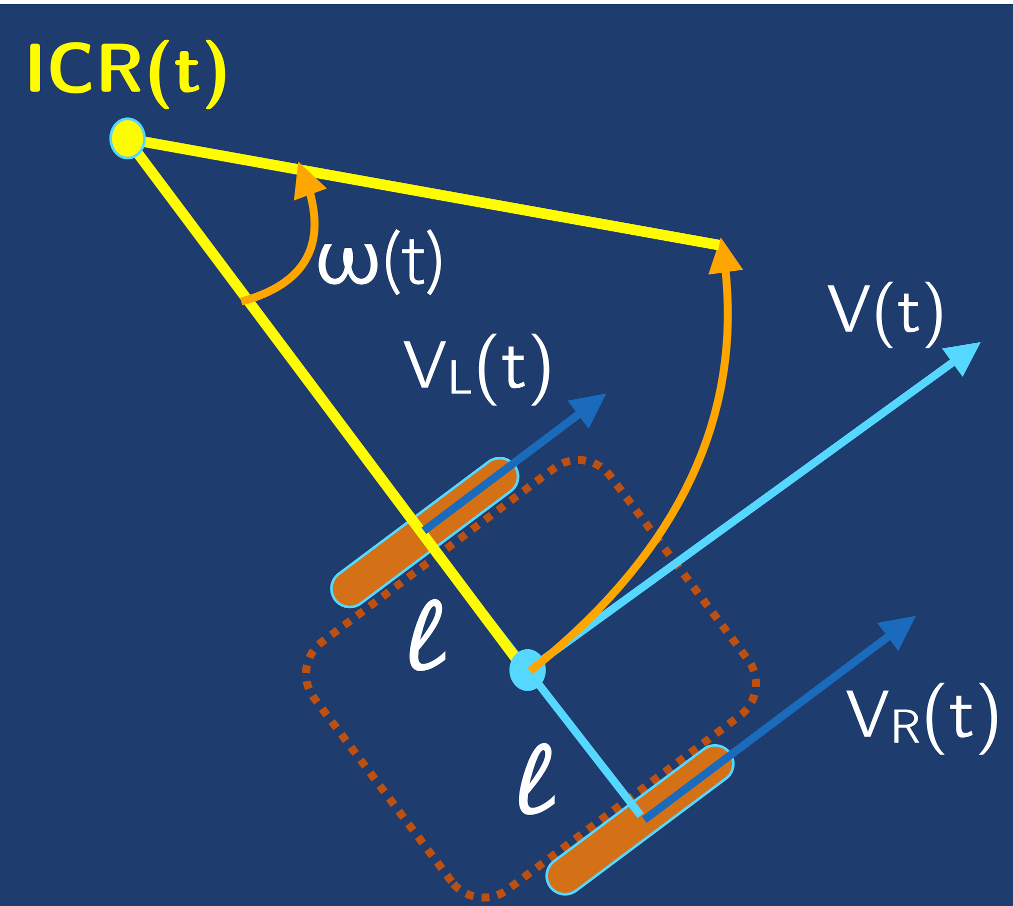
$v_C = \omega r_C \quad v_E = 2 v_C$

$\vec{v}_P = \vec{v}_R + \vec{\omega} \times \vec{r}_{RP}$

- Wheel's motion can instantaneously be seen as a pure rotation about an axis, normal to the plane of motion, the **axis of instantaneous rotation**, or **of zero velocity**. The point where the axis intersects the plane of motion is the ICR
- $\rightarrow$ Rigid body's motion happens along a **circumference centered in the ICR,** that has **zero velocity**
- The farther the distance from the ICR, proportionally the larger is the velocity

FROM WHEELS TO ROBOT CHASSIS

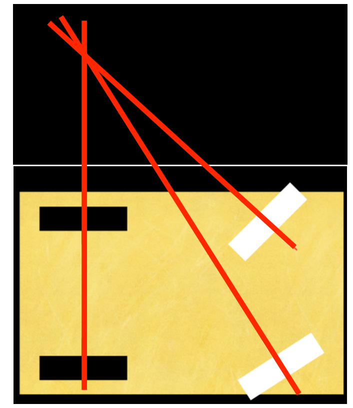




- <u>No side-motion constraints:</u> no motion along the line ⊥ to the plane of each wheel
- For each wheel, (**v** — **ICR**) vector overlaps with the no side-motion line
- **At any time t, ICR is the intersection of all zero motion lines from wheels**

ROBOT'S INSTANTANEOUS CENTER OF ROTATION

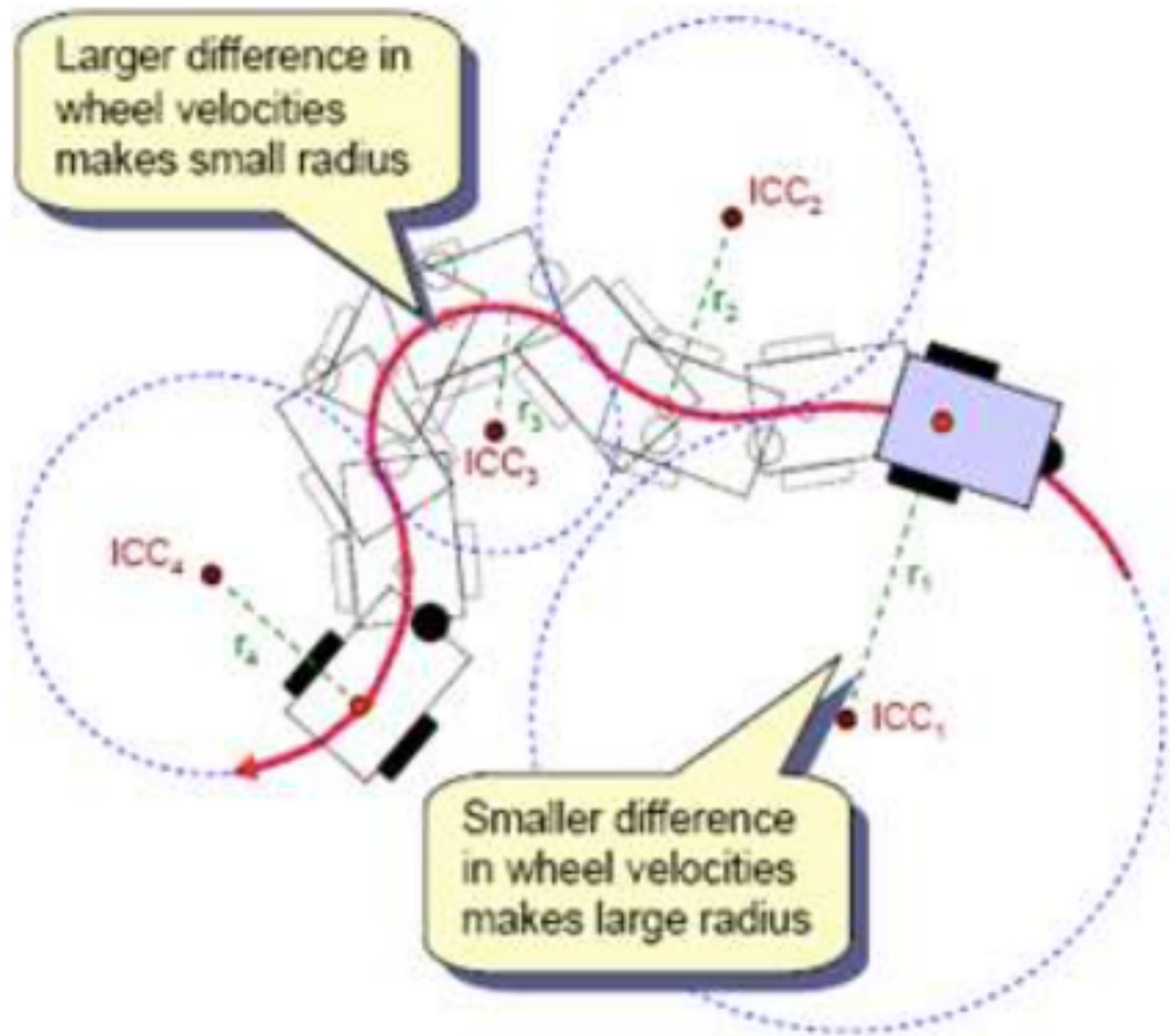
- The ICR is the point around which each wheel makes a circular course, with a different radius, depending on wheel's position on the chassis



- ICR defines a zero motion line drawn through the horizontal axis perpendicular to the plane of each wheel on the chassis

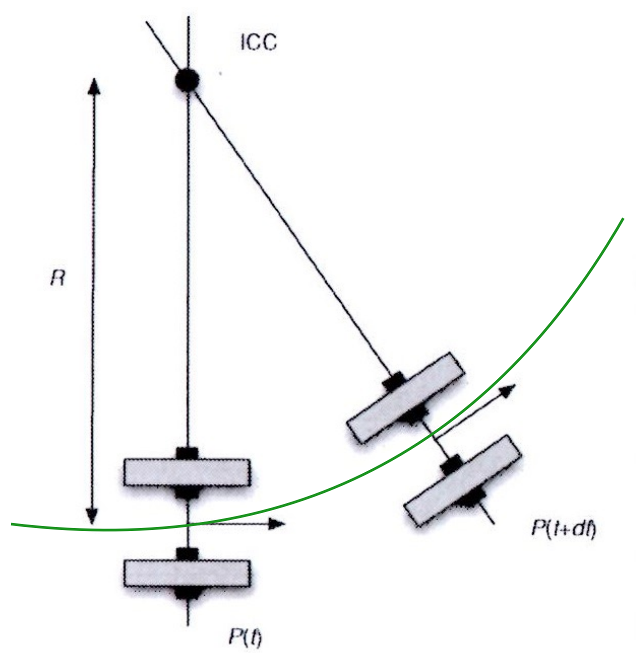
- At any time t, the robot reference point (between the wheels in the figure) moves along a circumference of radius R with center on the zero motion line, the center of the circle is the ICR

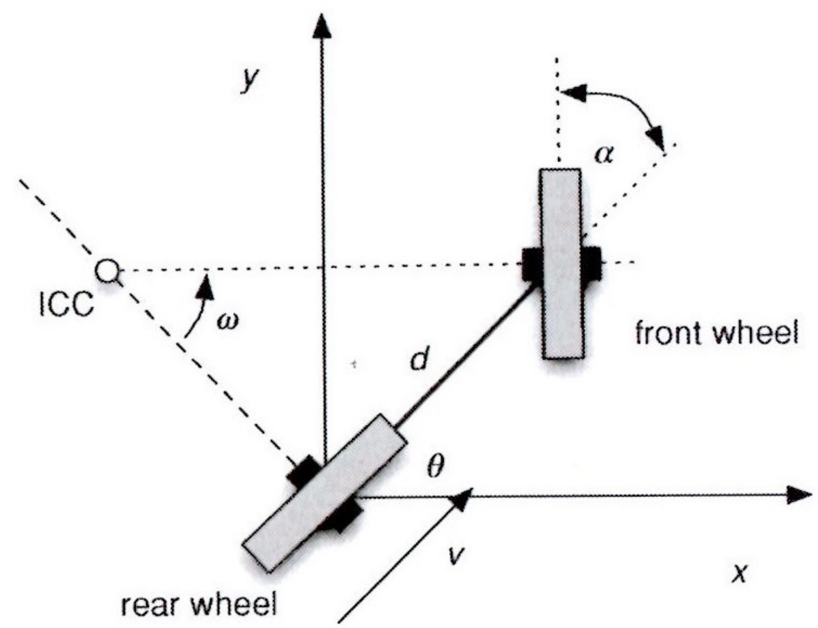
- The ICC changes over time as a function of the individual wheel velocities, and, in particular , of their relative difference

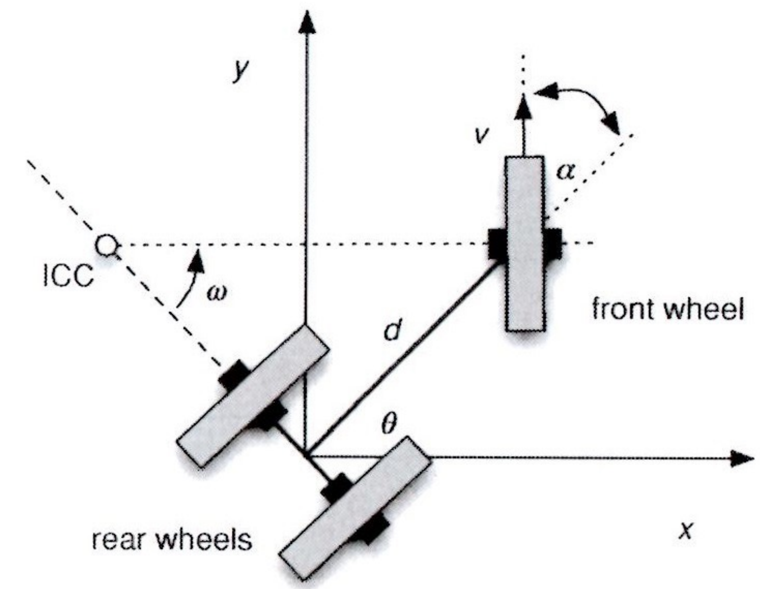


ICC FOR DIFFERENT DRIVING MODES

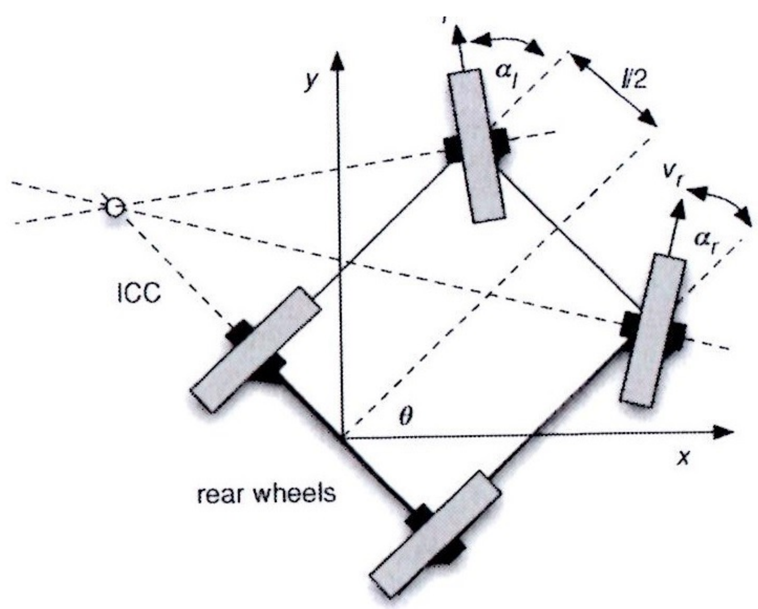
The position of the ICC depends on the instantaneous wheels' motion, that determines the instantaneous angular velocity ω of the robot around the ICC



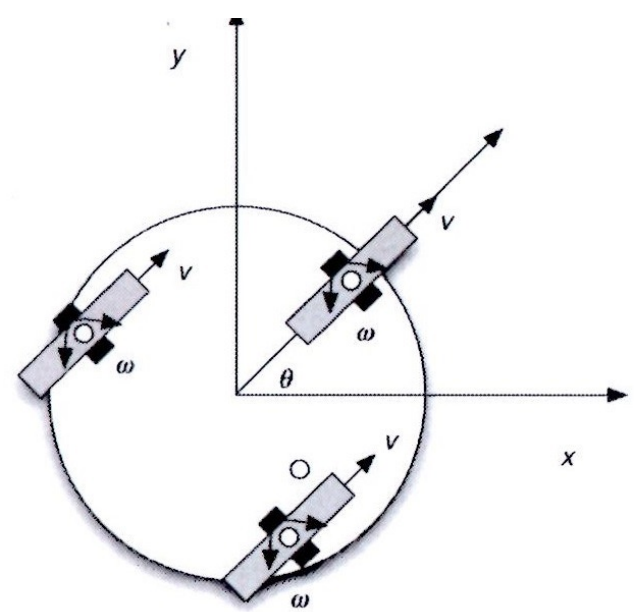




(a) Tricycle schematic



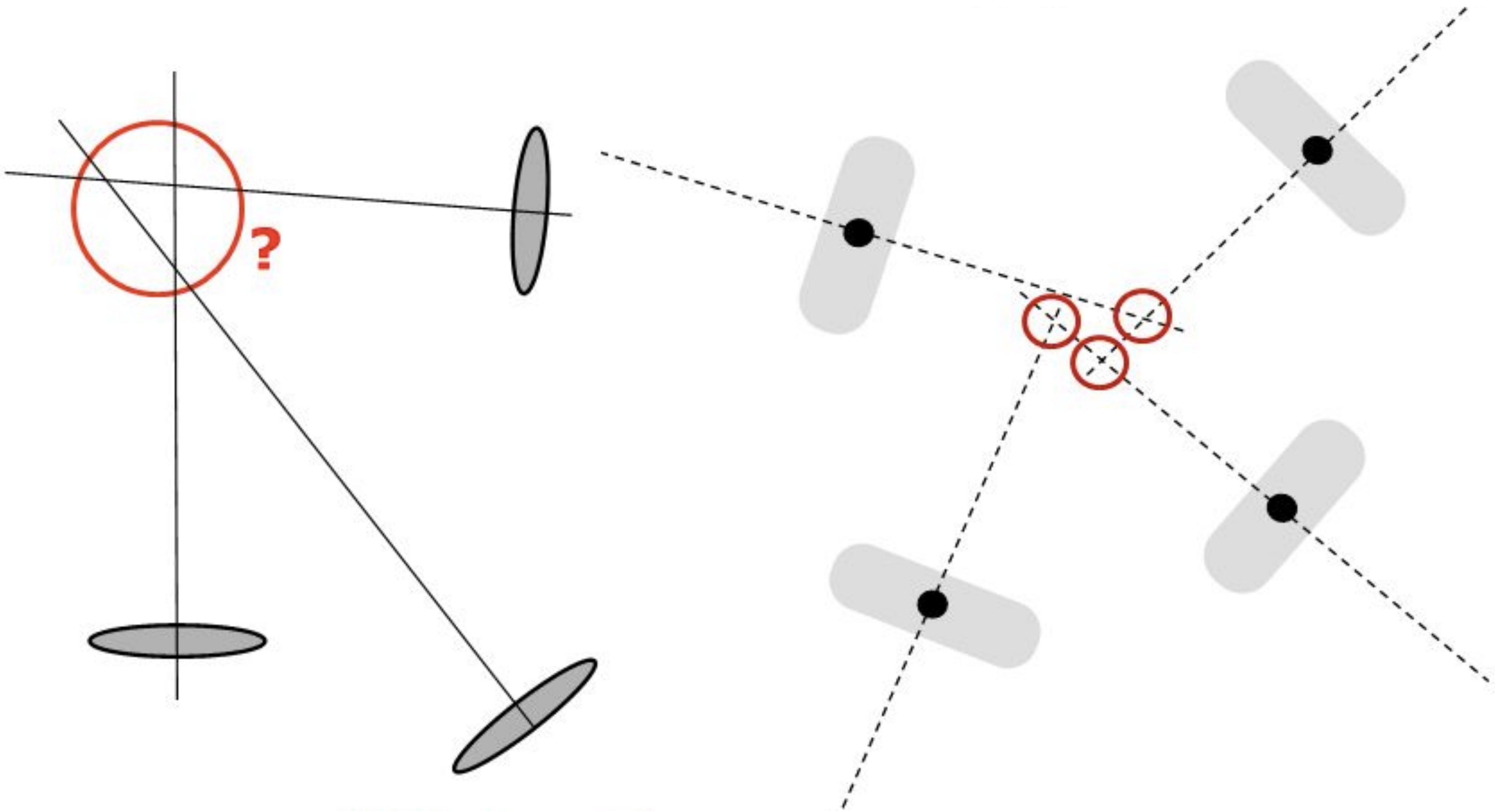
(a) Ackerman steering



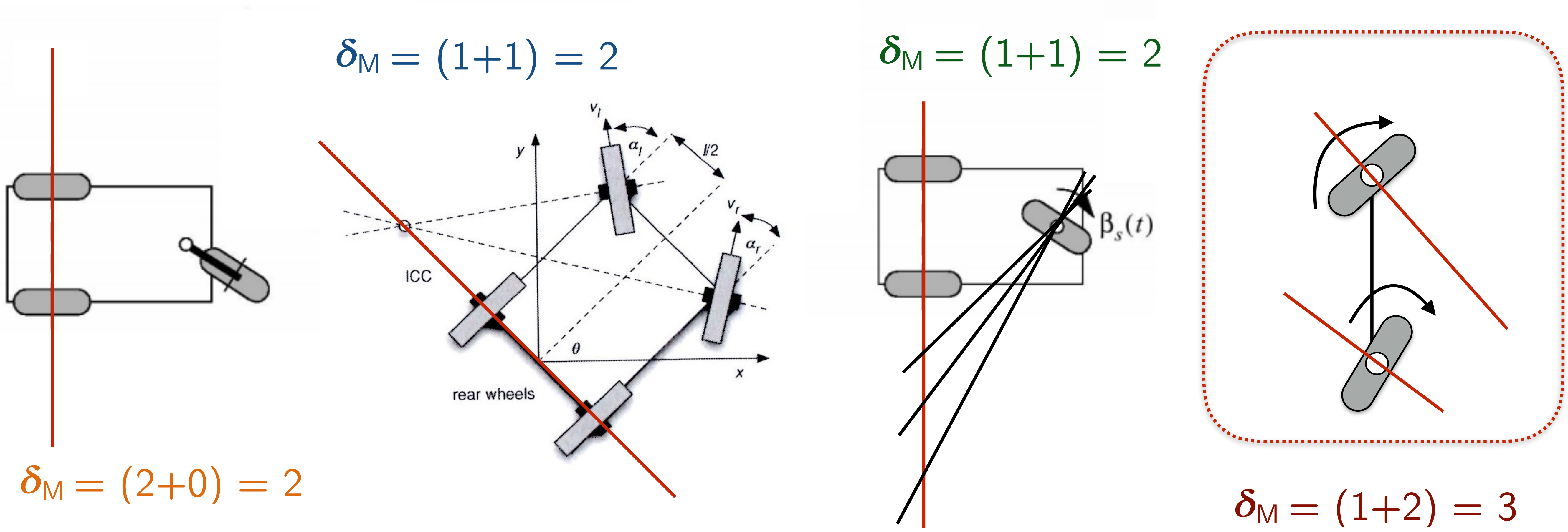
(a) Synchronous drive

For a holonomic robot the ICC it's in the center of the robot

NO ICC, NO MOTION (WITHOUT SLIPPAGE)



MOBILE ROBOT MANEUVERABILITY AND ICC/ICR



- In the first three cases, the ICR cannot range anywhere on the plane, but it must lie on a predefined line with respect to the robot reference frame

- For any robot with $\delta_M = 2$, the ICR is always constrained on a line
- For any robot with $\delta_M = 3$, the ICR can be set to any point on the plane

Let's sum up all notions and results so far:

- **Maneuverability (δ_M):** # of *control* degrees of freedom for realizing motion (changing its pose) that a robot has available
- Motion degrees of freedom can be manipulated **directly (δ_m)**, through wheels' velocity, and **indirectly**(δ_s) through steering configurations and moving
- **Configuration space $\mathscr{C}$:** the space of the *m*-dimensional generalized configuration coordinates representing all possible robot configurations (robot's structure + environment)
- **DOFs of the robot:** # of independent coordinates (out of *m*) of the configuration space $\rightarrow$ # of parameters the robot can independently act upon to change its configuration (e.g., x,y,θ), which depends on the presence or not of geometric / holonomic constraints
- **DOFs of the workspace $\mathscr{W}$:** DOFs (# of independent coordinates) of the embedding operational environment that the robot can reach (e.g., 3 DOFs for a robot in 2D space)
- DOF(workspace) $\gtreqless$ DOF(robot)
- How the robot is able to move from one configuration to another in the configuration space? What type of paths are possible? What type of trajectories?
- We need to relate maneuverability to DOFs .... $\rightarrow$

MANEUVERABILITY, DOF, NON HOLONOMIC ROBOT

- **Generalized velocity space $\mathscr{V}$:** the m-dimensional space of the time derivatives of the generalized coordinates of the configuration space (e.g., dx/dt, dy/dt, dθ/dt)
- **DOFs of the generalized velocity space:** # of independent velocity coordinates (out of m) of the generalized velocity space $\rightarrow$ # of independent velocity parameters that the robot can control to change its motion, which depend on the presence or not of kinematic / non holonomic constraints
- **Admissible velocity space**: given the kinematic constraints, the n-dimensional subspace of $\mathscr{V}$ ($n \leq m$) that describes the independent components of motion that the robot can directly control through wheels' velocities
- **Differential degrees of freedom (DDOF):** The number n of dimensions in the velocity space of a robot $\rightarrow$ the number of independently achievable velocities

$$\text{DDOF} = \delta_m \qquad \text{DDOF} \leq \delta_M \leq \text{DOF}$$

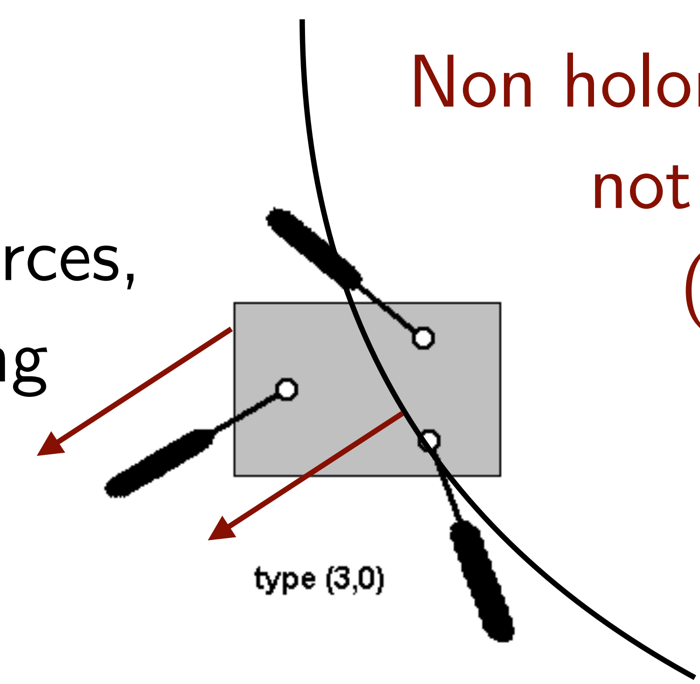
- **DOF** governs the robot's ability to achieve various poses in $\mathscr{C}$
- **DDOF** governs a robot's ability to achieve various paths in $\mathscr{C}$

HOLONOMIC ROBOTS

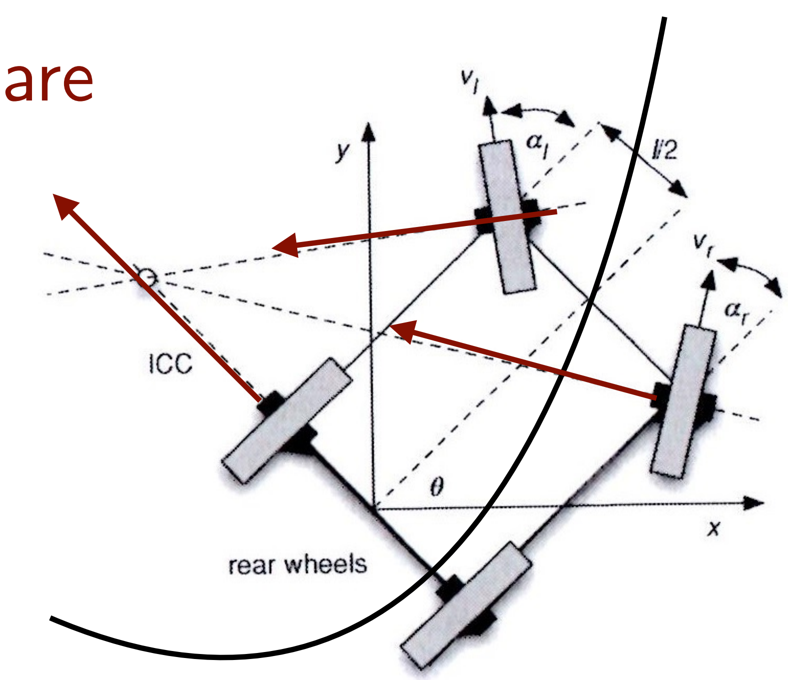
Holonomic robot: Iff the controllable degrees of freedom are equal to total degrees of freedom: DDOF = DOF($\mathscr{W}$)

- An holonomic robot can directly control all velocity components
- The presence of kinematic constraints reduces the capability to freely execute paths and decreases the DDOFs, making them less than DOFs
- An omnidirectional robot, that has no kinematic constraints (no standard wheels), is an example of holonomic robot: $\boldsymbol{\delta_M = 3 + 0 = \mathrm{DDOF} = \mathrm{DOF}}$

Lateral forces, skidding



Non holonomic constraints are not necessarily *bad* (for stability)



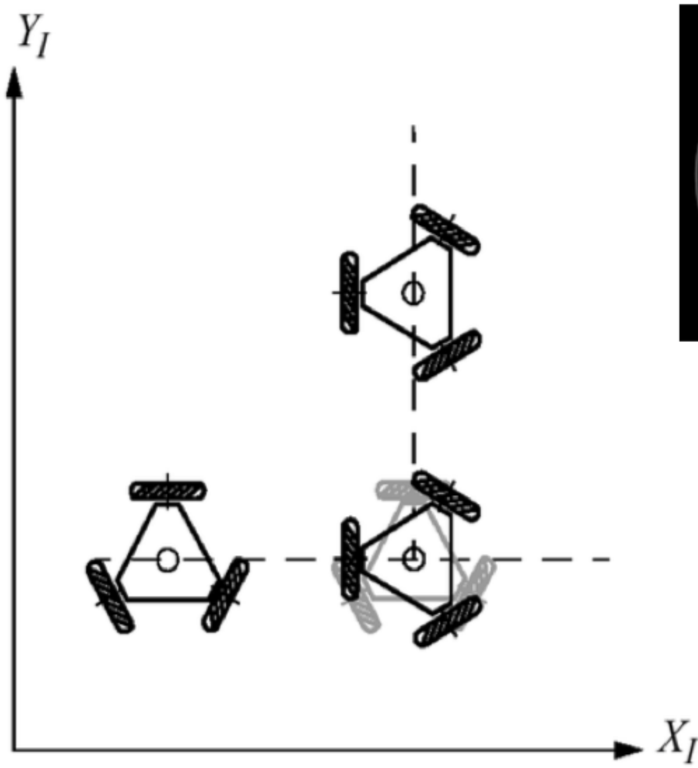
DEGREE OF MANEUVERABILITY VS. DOFS

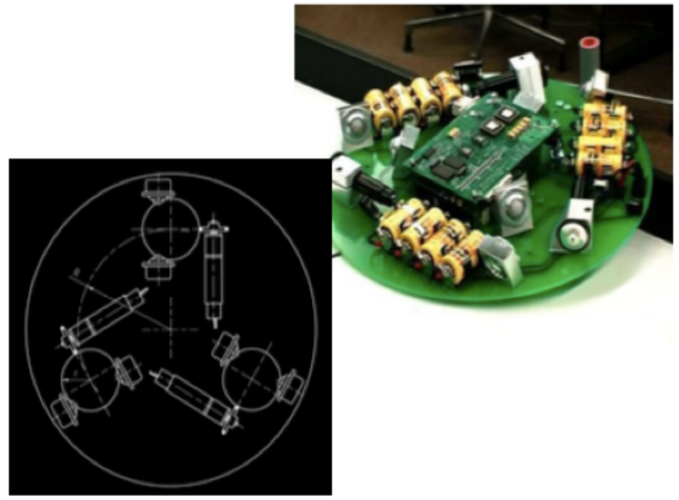
What about steering freedom?

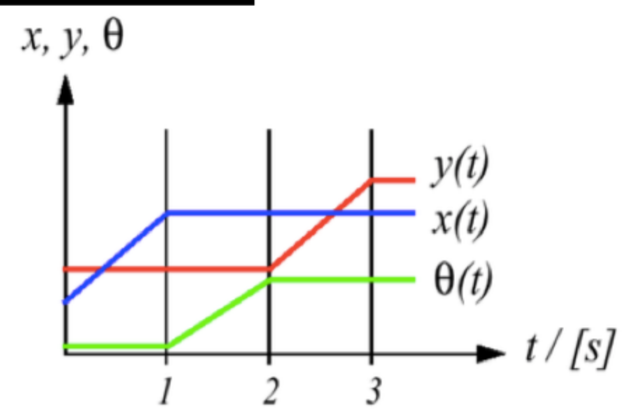
- $\delta_M = 3 \Rightarrow$ **ability to freely manipulate the ICR**
- Doesn't this mean that the robot is unconstrained selecting its paths?
- Yes! But $\boldsymbol{\delta_M = 3 + 0 \neq 1 + 2}$ (e.g., two-steer bicycle)
- This has an impact in the context of trajectories rather than paths
- Trajectory = path + time (m+1 dimensions)

Omni vs. Two-steer making *trajectories* …

TRAJECTORY MAKING

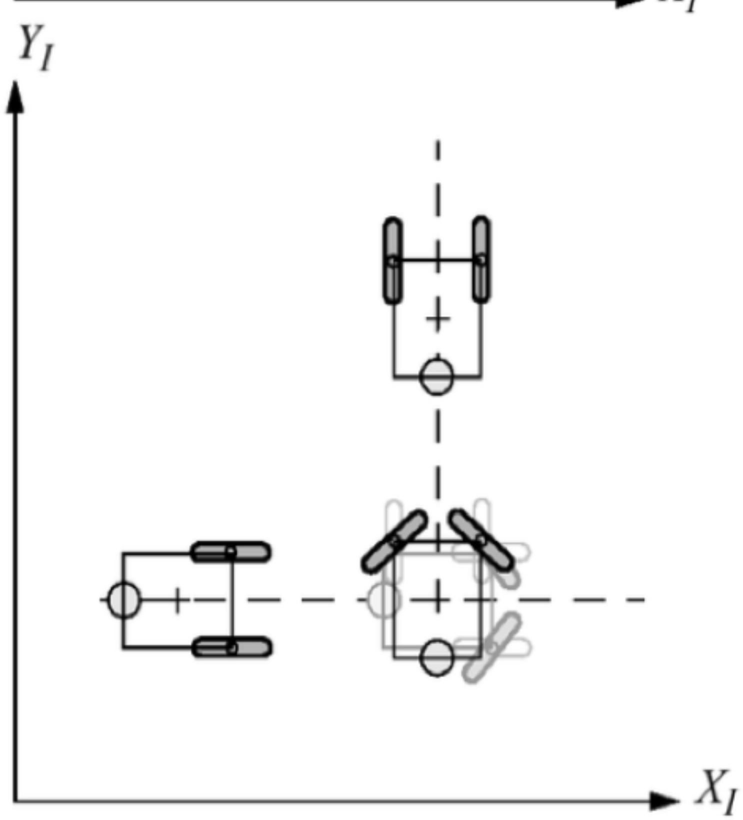


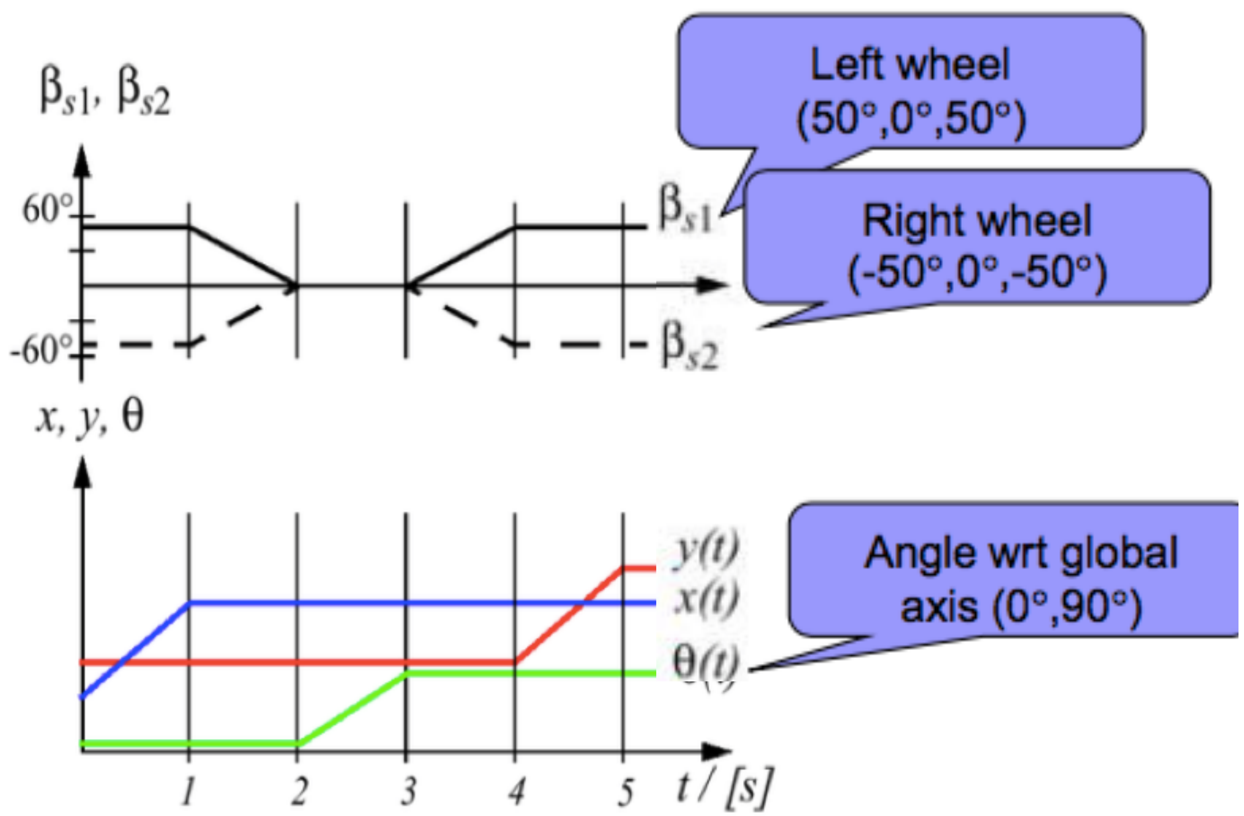




- A robot has a goal trajectory in which the robot moves along axis X_I at a constant speed of 1 m/s for 1 second.
- Wheels adjust for 1 second. The robot then turns counterclockwise at 90 degrees in 1 second.
- Wheels adjust for 1 second. Finally, the robot then moves parallel to axis Y_I for 1 final second.

acceleration $= \infty$





Arbitrary trajectories are not attainable!

(changes to internal DOFs are required and take time)

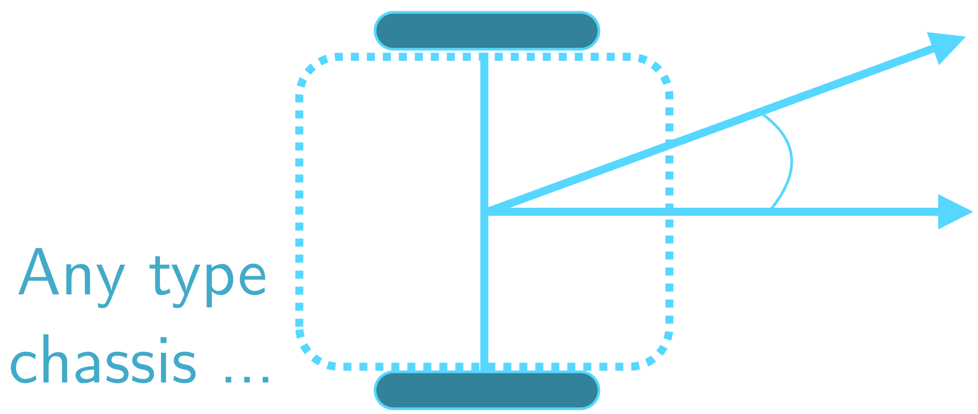
DOFS FOR DIFFERENT ROBOTS

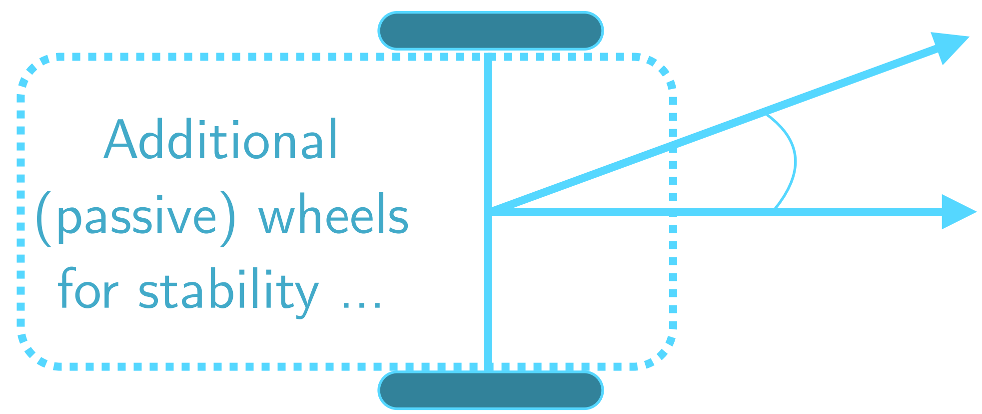
	dim $\mathcal{C}$	Degrees of freedom	Number of actuators	Actuation	Rolling constraints	Holonomic
Train	1	1	1	full		✓
2-joint robot arm	2	2	2	full		✓
6-joint robot arm	6	6	6	full		✓
10-joint robot arm	10	10	10	over		✓
Hovercraft	3	3	2	under		
Car	3	2	2	under	✓	
Helicopter	6	6	4	under		
Fixed wing aircraft	6	6	4	under		
DEPTHX AUV	6	6	6	full		✓

DIFFERENTIAL (*) VEHICLES

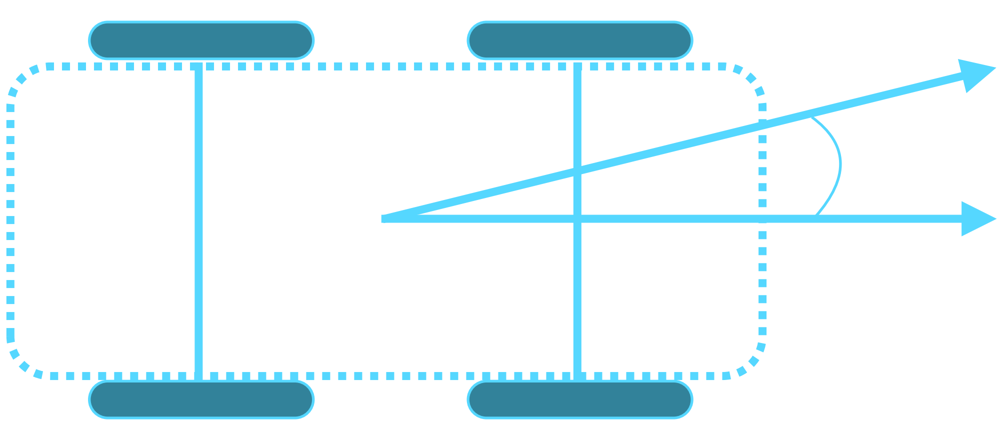
Differential steering (vehicle, robot)

two standard wheels mounted on a single axis are independently powered and controlled, providing both *drive* and *steering* functions through the *motion difference* between the wheels





total wheel pairs can be more than two, making control more complex

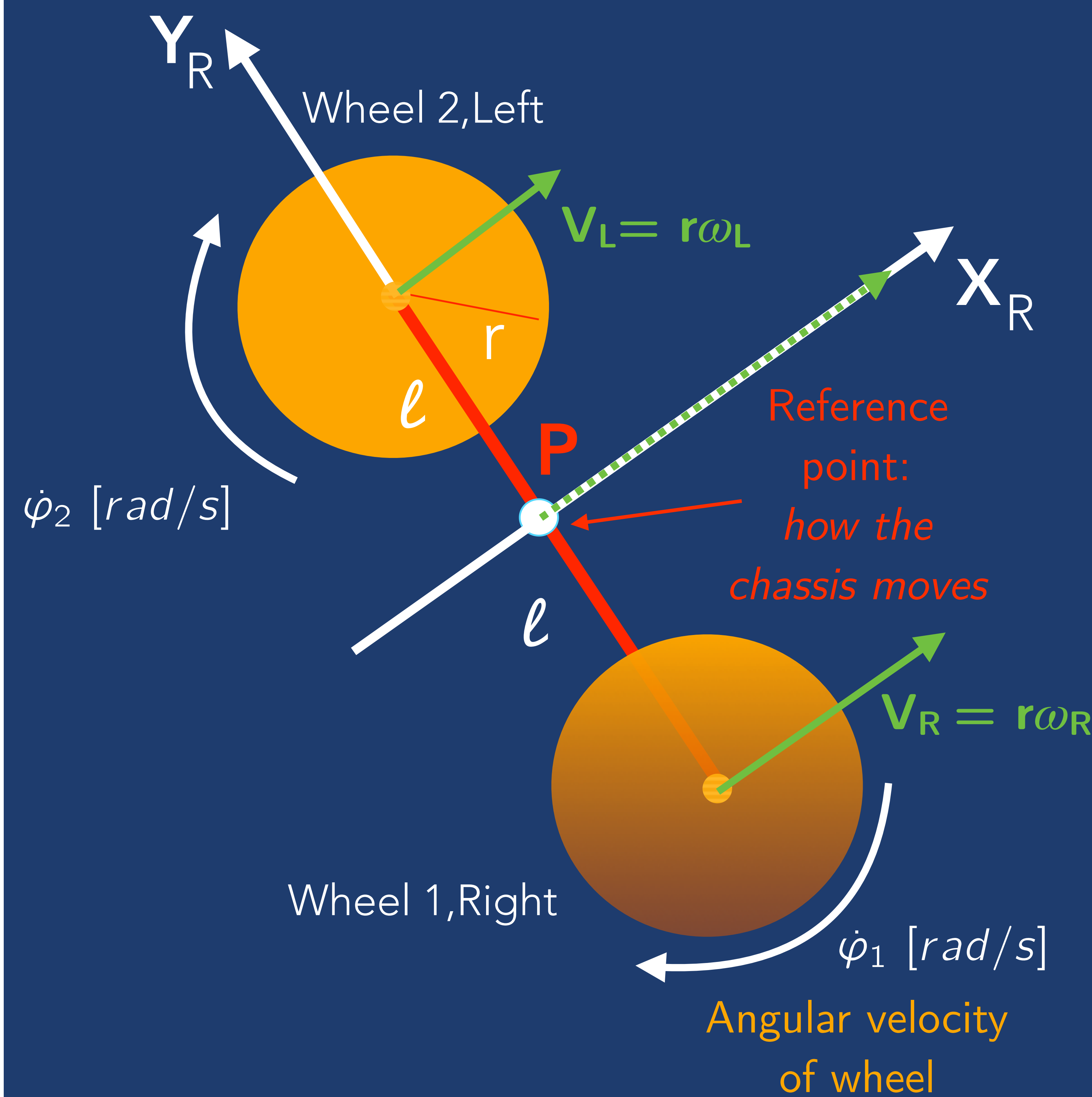


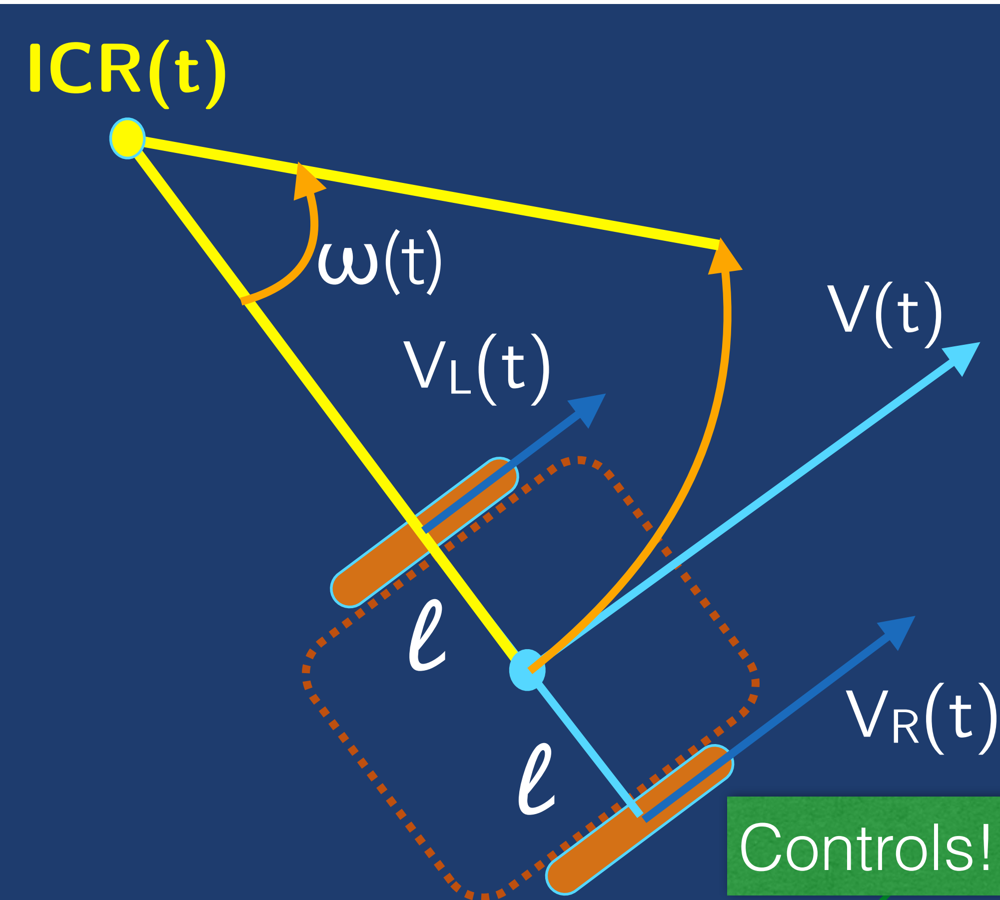
What are the kinematic equations?

Differential drive

In automotive engineering, it refers to the presence of a differential gear or related device to transfer different motion to the steering wheels on a same axis (e.g., frontal wheels of a normal car)

FROM WHEELS TO ROBOT CHASSIS





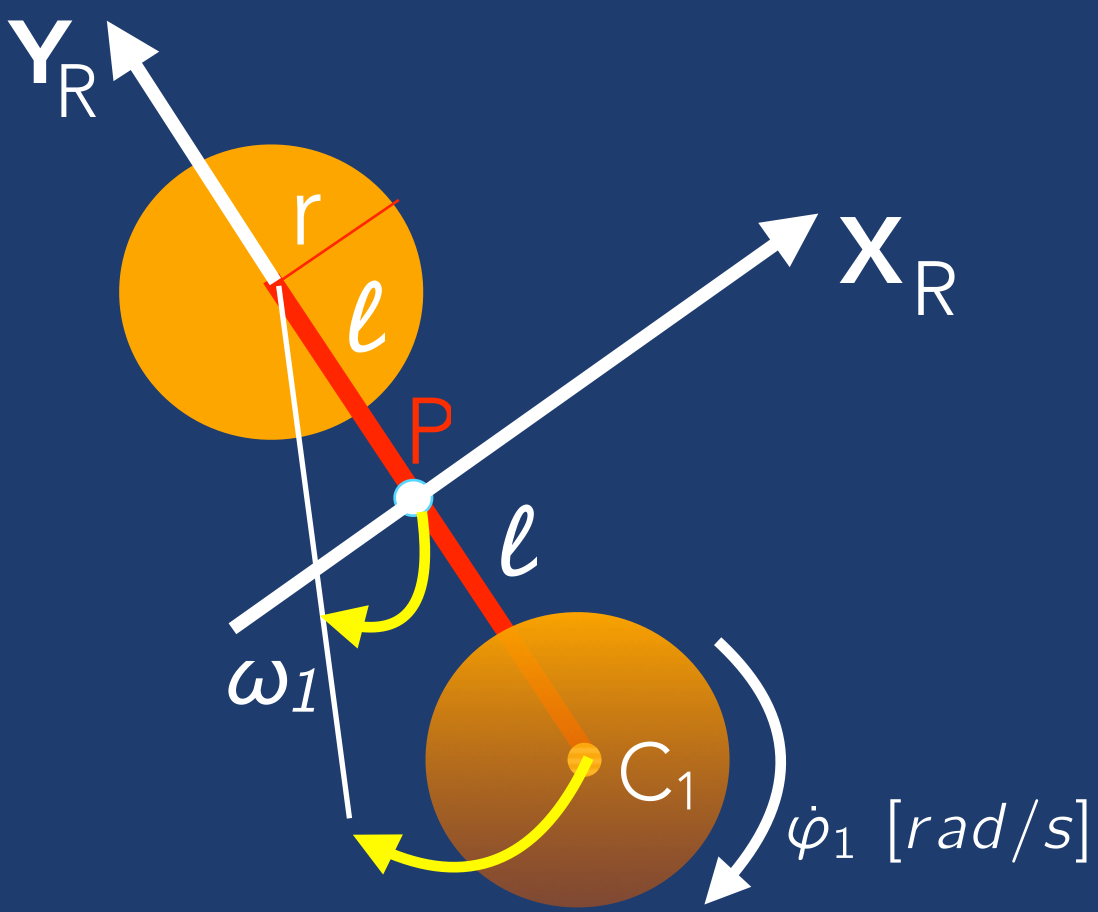
$$\omega(t)(R(t)+\ell) = V_R(t)$$
$$\omega(t)(R(t)-\ell) = V_L(t)$$

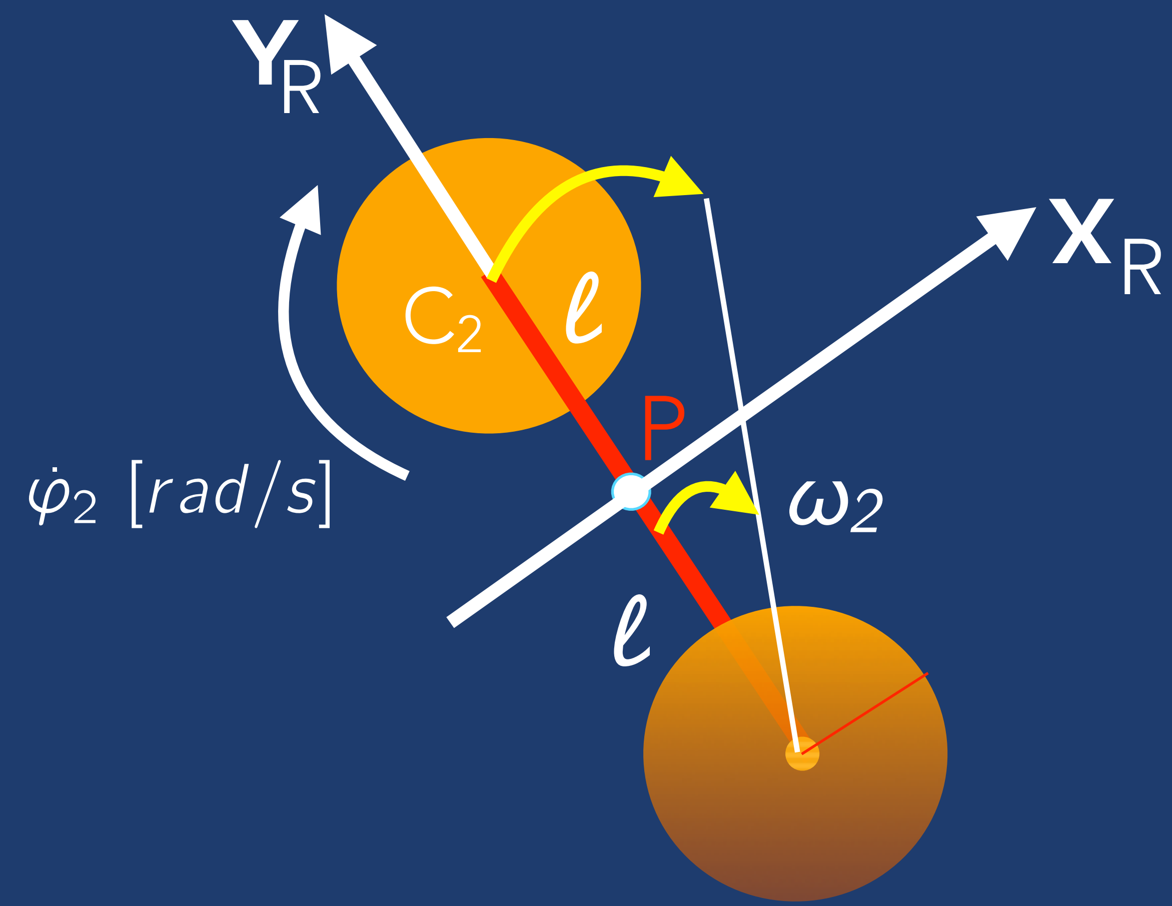
At any specific time instant *t:*

$$R(t) = \ell\,\frac{V_R(t)+V_L(t)}{V_R(t)-V_L(t)}$$

$$\omega(t) = \frac{V_R(t)-V_L(T)}{2\ell}$$

COMPOSITION OF ANGULAR VELOCITIES





If only the right, C_1 wheel spins (forward), the contribution to the angular velocity of P:

$$\omega_1 = \frac{r\dot{\varphi}_1}{2l}$$

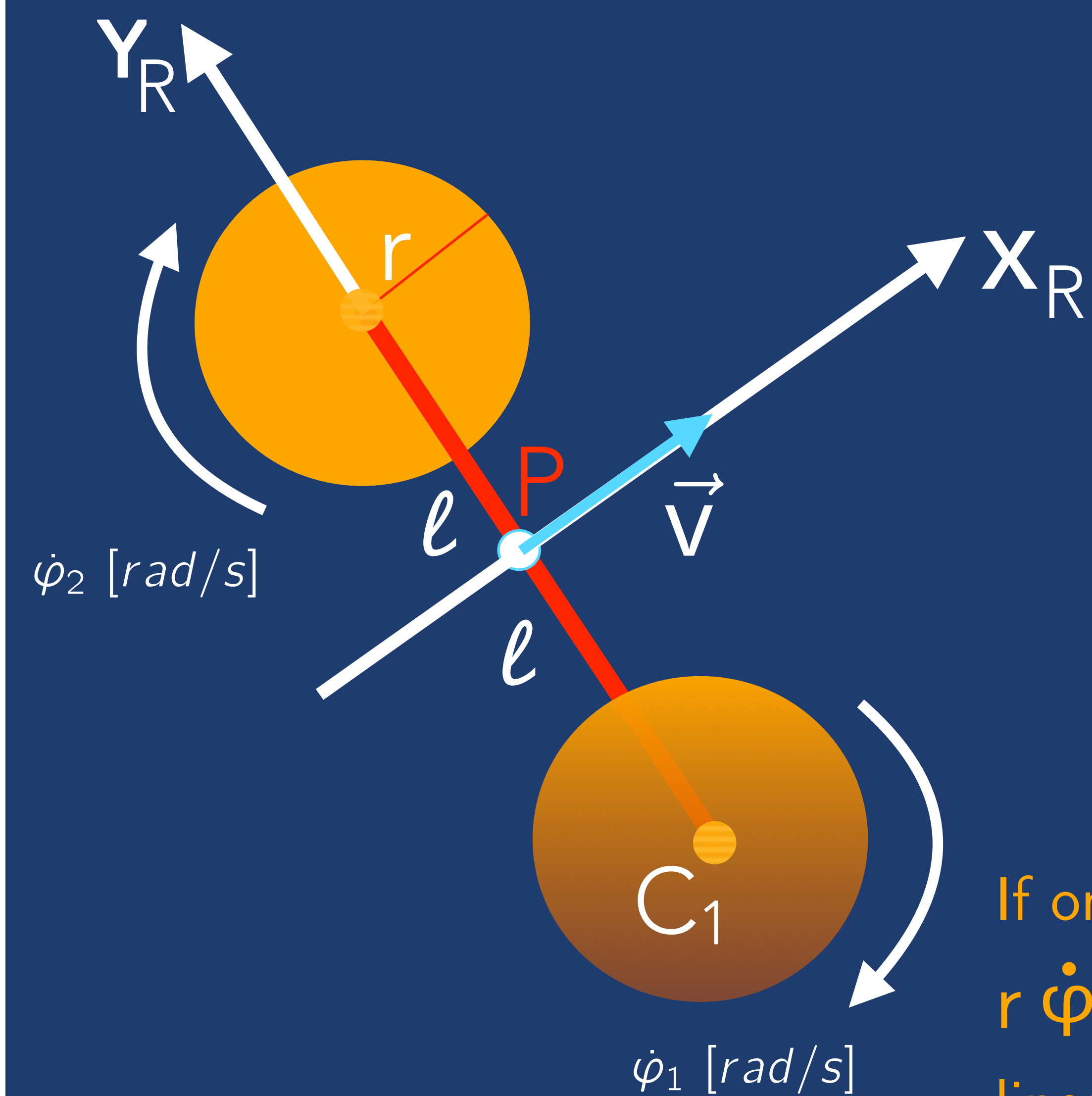
If only the left, C_2 wheel spins (forward), the contribution to the angular velocity of P:

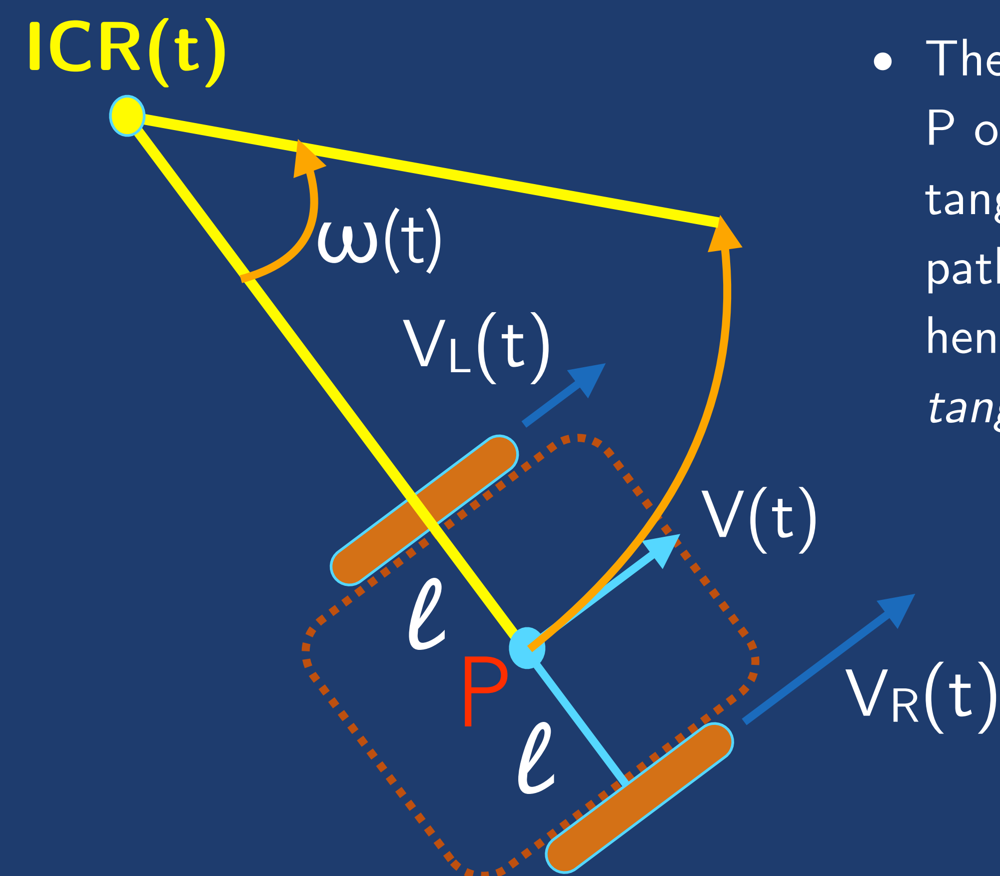
$$\omega_2 = -\frac{r\dot{\varphi}_2}{2l}$$

The contributions of each wheel to the angular velocity in P can be ***computed independently and added up (signed)***

$$\omega_P = \frac{r\dot{\varphi}_1 - r\dot{\varphi}_2}{2\ell}$$

COMPOSITION OF LINEAR VELOCITIES





- The linear velocity of a point P on the rigid chassis is tangential to the circular path followed by the point, hence, is also termed a *tangential* velocity

If only the right wheel spins (forward), the linear velocity of C_1 is $r\,\dot{\varphi}_1$, that of C_2 is 0, and that of P is half of that of C_1, since linear velocity scales linearly with the radius (centered in C_2). An analogous reasoning applies when C_2 is the only spinning wheel

The contributions of each wheel to the tangential velocity in P can be ***computed independently and added up, each divided by 2***

$$v_P = \frac{r\dot{\varphi}_1 + r\dot{\varphi}_2}{2}$$

SPECIAL CASES FOR DIFFERENTIAL MOTION

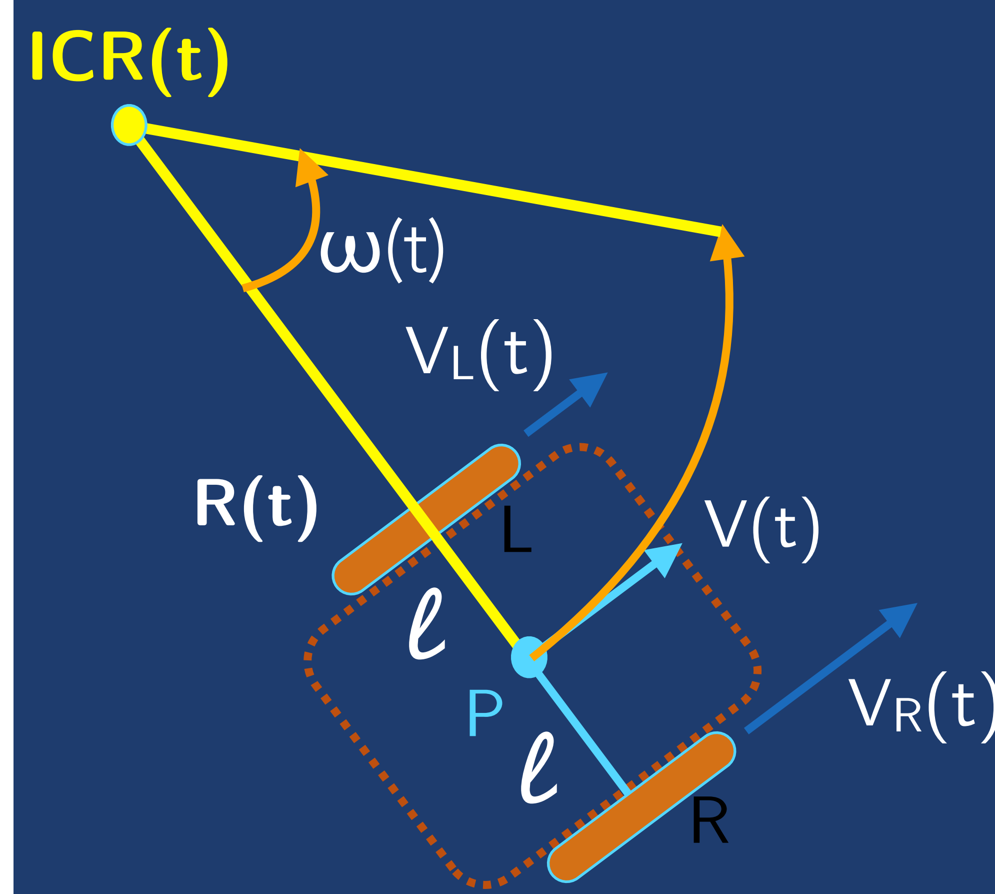
- **$V_L = V_R$** ➔ $R = \infty$, and there is effectively no rotation, $\omega = 0$: Forward linear motion in a straight line

- **$V_L = -V_R$** ➔ $R = 0$, meaning that it coincides with P, and $\omega = -V/\ell$: Rotation about the midpoint of the wheel axis (in place rotation)

- **$V_L = 0$** ➔ $R = \ell$ (in the center of L), $\omega = V_R/2\ell$: Counterclockwise rotation about the left wheel

- **$V_R = 0$** ➔ $R = -\ell$ (in the center of R), $\omega = -V_L/2\ell$: Clockwise rotation about the right wheel

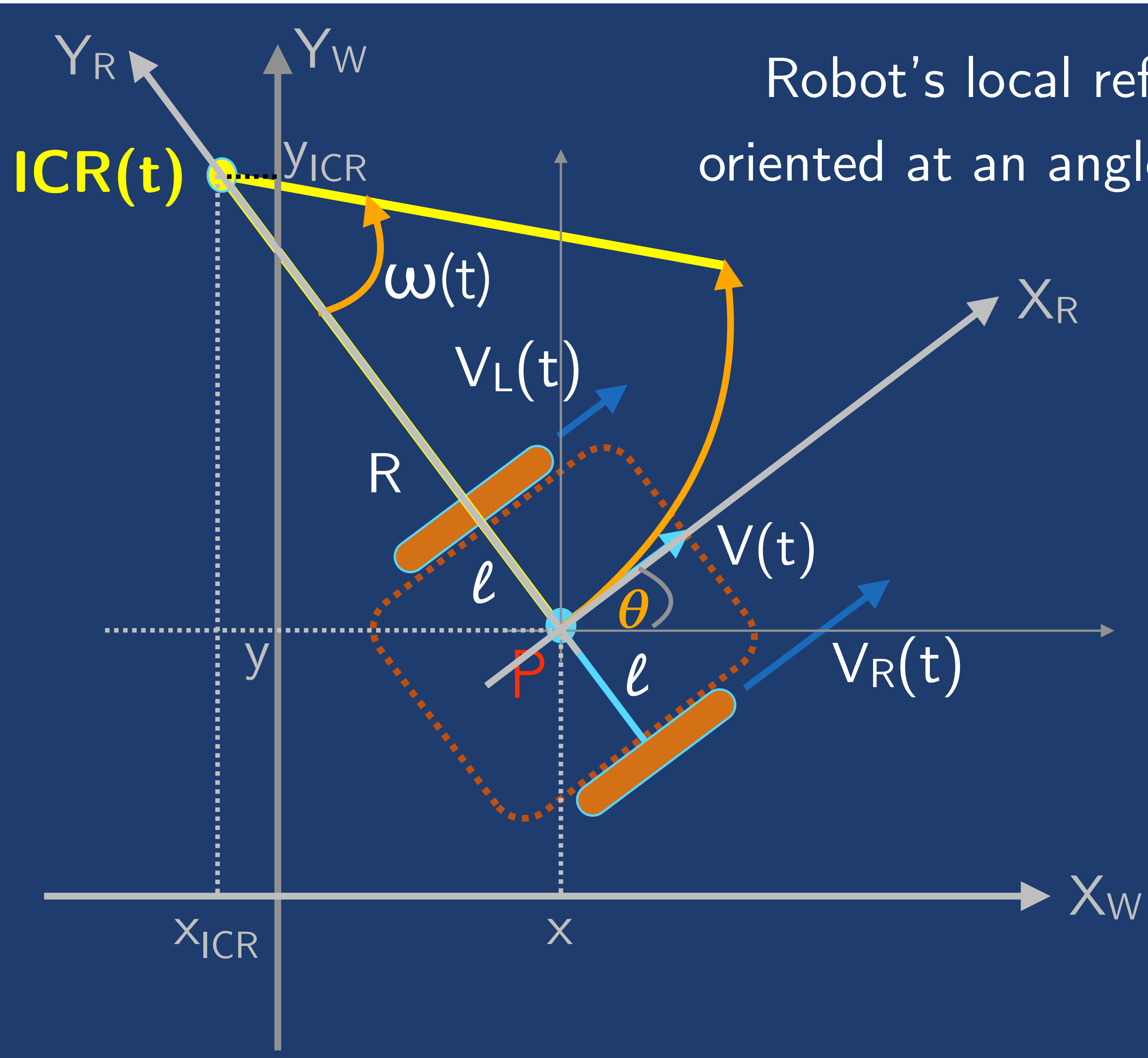
Do you spot any potential practical issues?



$$R(t) = \ell \, \frac{V_R(t)+V_L(t)}{V_R(t)-V_L(t)}$$

$$\omega(t) = \frac{V_R(t)-V_L(T)}{2\ell}$$

REFERENCE FRAMES AND POSITION OF THE ICR



Robot's local reference frame {R} is in coord (x,y) and oriented at an angle θ wrt to the world reference frame {W}

$$^W ICR = {}^W\xi_R \cdot {}^R ICR$$

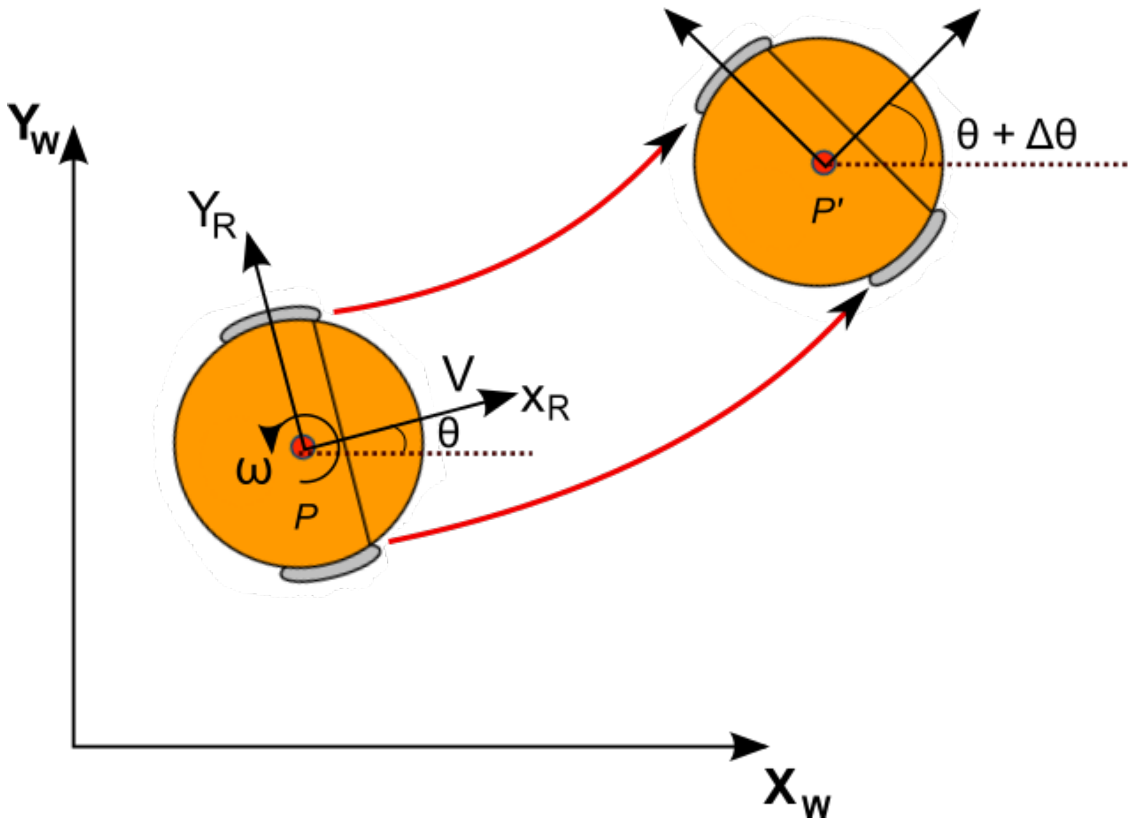
$$^R ICR = [0 \; R]^T$$

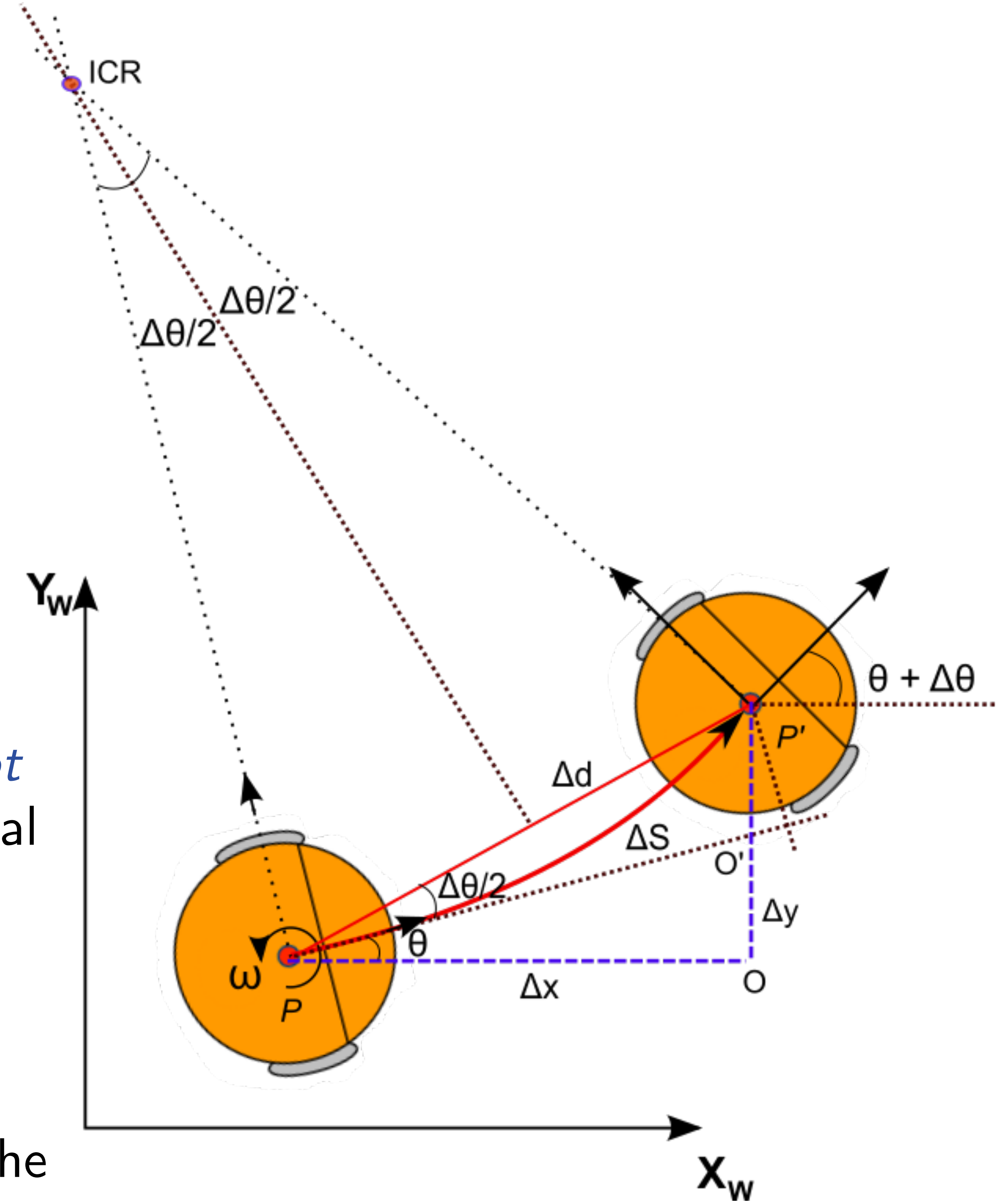
$$^W\widetilde{ICR} = {}^W\mathbf{T}_R \, {}^R\widetilde{ICR}$$

$$\begin{bmatrix} x_{ICR} \\ y_{ICR} \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ R \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{ICR} \\ y_{ICR} \end{bmatrix} = \begin{bmatrix} x - R\sin(\theta) \\ y + R\cos(\theta) \end{bmatrix}$$

ROBOT POSE EVOLUTION AS A FUNCTION OF ICR





At a time t, an instantaneous motion of duration δt results in an infinitesimal change in orientation equal to $\Delta\theta$, and in an infinitesimal displacement ΔS:

what is the robot pose ${}^{W}\xi_R$ at time $(t + \delta t)$?

The ICR will not change, and the new pose is the result of a rotation $\Delta\theta = \omega\delta t$ of the robot about the ICR (ω is constant during the infinitesimal interval)

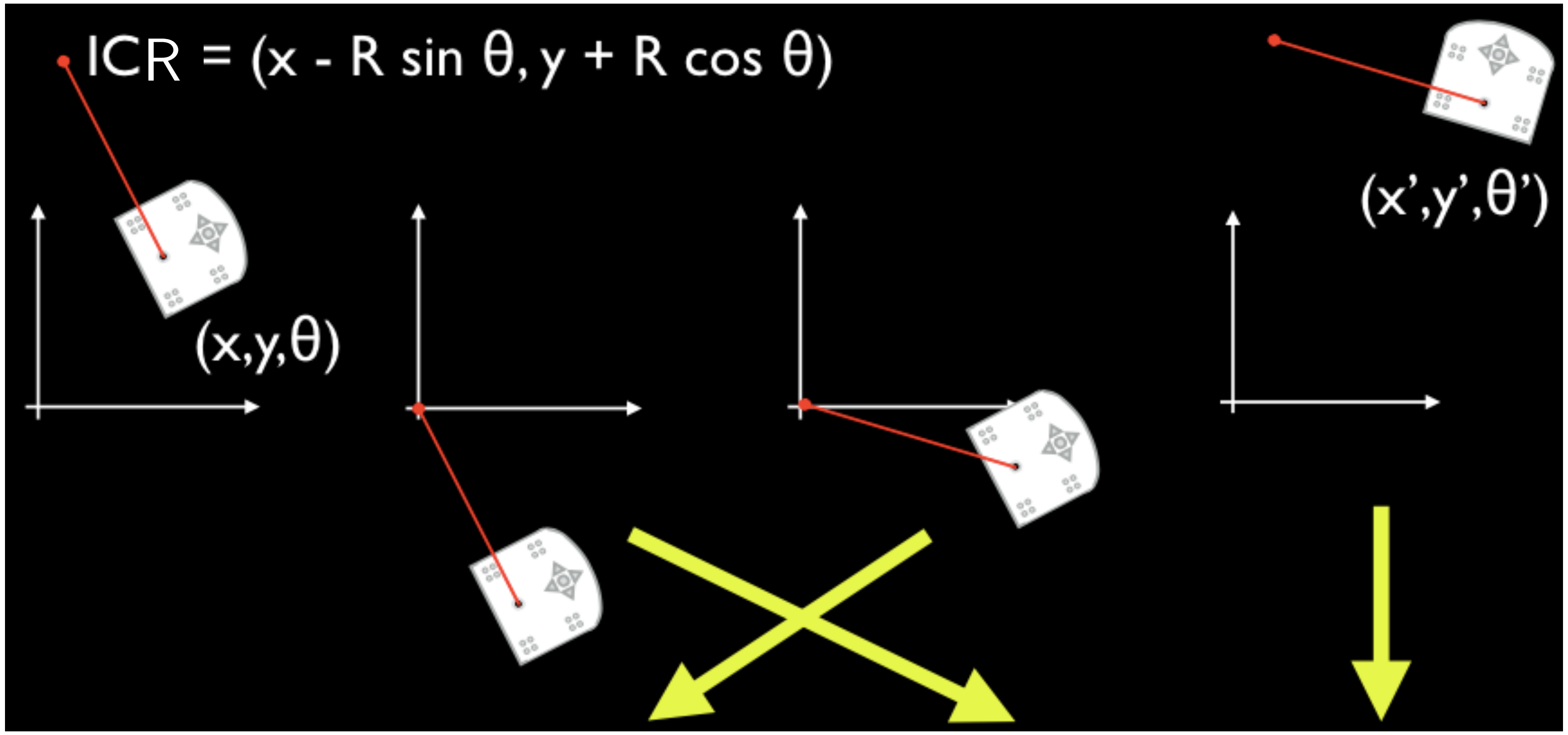
(1) translation of the ICR at {W} origin, (2) rotation of $\Delta\theta$, (3) translation back to the ICR

MOTION EQUATIONS FOR A ROBOT ROTATING ABOUT ITS ICR

Motion of a robot rotating a distance R about its ICR with an angular velocity of ω

(1) translation of the robot, positioning the ICR at {W} origin
(2) rotation in place of $\Delta\theta = \omega\delta t$
(3) translation back of the ICR at its initial position

Equation valid for any mobile robot!



$$^{W}\begin{bmatrix} x(t+\delta t) \\ y(t+\delta t) \\ \theta(t+\delta t) \end{bmatrix} = \begin{bmatrix} \cos(\omega\delta t) & -\sin(\omega\delta t) & 0 \\ \sin(\omega\delta t) & \cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) - x_{ICR}(t) \\ y(t) - y_{ICR}(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} x_{ICR}(t) \\ y_{ICR}(t) \\ \omega\delta t \end{bmatrix}$$

Based on the velocity inputs to the right and left wheels, robot's pose can be computed

FORWARD KINEMATICS EQUATIONS

$$
{}^{W}\begin{bmatrix} x(t+\delta t) \\ y(t+\delta t) \\ \theta(t+\delta t) \end{bmatrix} = \begin{bmatrix} \cos(\omega\delta t) & -\sin(\omega\delta t) & 0 \\ \sin(\omega\delta t) & \cos(\omega\delta t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) - x_{ICR}(t) \\ y(t) - y_{ICR}(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} x_{ICR}(t) \\ y_{ICR}(t) \\ \omega\delta t \end{bmatrix}
$$

$$
{}^{W}\begin{bmatrix} x(t+\delta t) \\ y(t+\delta t) \\ \theta(t+\delta t) \end{bmatrix} = \begin{bmatrix} \left(x(t) - x_{ICR}(t)\right)\cos(\omega\delta t) - \left(y(t) - y_{ICR}(t)\right)\sin(\omega\delta t) + x_{ICR}(t) \\ \left(x(t) - x_{ICR}(t)\right)\sin(\omega\delta t) + \left(y(t) - y_{ICR}(t)\right)\cos(\omega\delta t) + y_{ICR}(t) \\ \theta(t) + \omega\delta t \end{bmatrix}
$$

$$
= \begin{bmatrix} R(t)\sin\left(\theta(t)\right)\cos\left(\omega\delta t\right) + R(t)\cos\left(\theta(t)\right)\sin\left(\omega\delta t\right) + x(t) - R(t)\sin\left(\theta(t)\right) \\ R(t)\sin\left(\theta(t)\right)\sin\left(\omega\delta t\right) - R(t)\cos\left(\theta(t)\right)\cos\left(\omega\delta t\right) + y(t) + R(t)\cos\left(\theta(t)\right) \\ \theta(t) + \omega\delta t \end{bmatrix}
$$

$$
= \begin{bmatrix} x(t) + R(t)\Big(\sin\left(\theta(t) + \omega\delta t\right) - \sin\left(\theta(t)\right)\Big) \\ y(t) - R(t)\Big(\cos\left(\theta(t) + \omega\delta t\right) - \cos\left(\theta(t)\right)\Big) \\ \theta(t) + \omega\delta t \end{bmatrix}
$$

Function of end time of motion

Function of start time of motion

$$
= \begin{bmatrix} x(t) + R(t)\Big(\sin\left(\theta(t) + \Delta\theta(t+\delta t)\right) - \sin\left(\theta(t)\right)\Big) \\ y(t) - R(t)\Big(\cos\left(\theta(t) + \Delta\theta(t+\delta t)\right) - \cos\left(\theta(t)\right)\Big) \\ \theta(t) + \Delta\theta(t+\delta t) \end{bmatrix}
$$

FORWARD KINEMATICS EQUATIONS

$$
{}^{W}\begin{bmatrix} x(t+\delta t) \\ y(t+\delta t) \\ \theta(t+\delta t) \end{bmatrix} = \begin{bmatrix} x(t) + R(t)\Big(\sin\left(\theta(t)+\omega\delta t\right) - \sin(\theta(t))\Big) \\ y(t) - R(t)\Big(\cos\left(\theta(t)+\omega\delta t\right) - \cos(\theta(t))\Big) \\ \theta(t)+\omega\delta t \end{bmatrix} = \begin{bmatrix} x(t) + R(t)\Big(\sin\left(\theta(t)+\Delta\theta(t+\delta t)\right) - \sin(\theta(t))\Big) \\ y(t) - R(t)\Big(\cos\left(\theta(t)+\Delta\theta(t+\delta t)\right) - \cos(\theta(t))\Big) \\ \theta(t)+\Delta\theta(t+\delta t) \end{bmatrix}
$$

Function of the ICR

$$
= \boxed{\begin{bmatrix} x(t) + \dfrac{v(t)}{\omega(t)}\Big(\sin(\theta(t)+\Delta\theta(t+\delta t)) - \sin(\theta(t))\Big) \\ y(t) - \dfrac{v(t)}{\omega(t)}\Big(\cos(\theta(t)+\Delta\theta(t+\delta t)) - \cos(\theta(t))\Big) \\ \theta(t)+\omega(t)\delta t \end{bmatrix}}
$$

Function of the issued velocities

To obtain ***future poses over time-extended intervals,*** it is necessary to provide initial conditions, specify geometry parameters, assign the linear and angular velocity profiles v(t) and ω(t), and ***integrate over time*** (which might not be obvious/easy)

In the specific case of a ***two-wheeled differential robot***, v(t) and ω(t) at the reference point P on the chassis are functions of the Left and Right speeds issued to the Left and Right wheel, respectively:

$$
\omega_P(t) = \frac{r\dot{\varphi}_R - r\dot{\varphi}_L}{2\ell} = \frac{v_R(t) - v_L(t)}{2\ell}
$$

$$
v_P(t) = \frac{r\dot{\varphi}_R + r\dot{\varphi}_L}{2} = \frac{v_R(t) + v_L(t)}{2}
$$