G4ip: a contraction-free calculus

Constructive Logic (15-317)
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Lecture 11

Our improvements to the sequent calculus for proof search so far has consisted on removing duplications of main formulas and imposing an order on the application of rules based on *don't care* choices (those that make no difference on the success or failure of proof search). Nevertheless, we still have the annoying $\supset L$ rule which can loop if we are not smart about it. We will finally address this problem now.

1 G4ip (or LJT)

The *contraction-free* calculus for intuitionistic logic we show here was proposed by Roy Dyckhoff in 1992 [1]. The motivation for developing a contraction free calculus is two-fold: (1) avoid checks for loops during proof search, which can be expensive (remember how the proof of $\neg\neg(A \vee \neg A)$ can loop); and (2) prove the logic is *decidable* by showing a terminating proof search procedure.

The way to get rid of contraction on the $\supset L$ rule is to consider what is the format of the antecedent of the implication, i.e., what does A look like in $A \supset B$? Since A is a formula, it can be any of those:

- atom;
- conjunction;
- disjunction;
- implication;
- true; or
- false.

Let's analyse case by case.

1.1 Atom

If the antecedent of the implication is an atom, we have the following premises when applying $\supset L$:

$$\frac{\Gamma, a \supset B \to a \quad \Gamma, B \to C}{\Gamma, a \supset B \to C} \supset L$$

We analyse how the left premise can be continued.

1. If $a \in \Gamma$, then we can close this premise with $init$ and continue with the proof of the right premise.

2. We can apply the $\supset L$ rule again, which would result on another premise $\Gamma, a \supset B \to a$. But this option is useless... Remember that we only needed the repetition of the implication formula on the left if we happened to extract a different information from A in the meantime. Since our A is an atom, there is nothing new to be obtained from it, and applying $\supset L$ again would only get us to the same problem. So in the case the antecedent is an atom, that's a non-option.

3. A rule is applied to another formula in Γ. In this case, it is either an invertible rule, which can just as well be applied eagerly before this $\supset L$, or a non-invertible rule. From our set of non-invertible rules, we know that this can only be another $\supset L$ since we cannot apply a disjunction right rule to an atom. Turns out the implication left will permute down the other implication left (we leave this as an exercise). We conclude thus that every rule on the left branch can be permuted down, on which case we can delay the application of $\supset L$ to $a \supset B$ until we can actually close the left branch (first case).

This analysis suggests we can have only the following instance:

$$\frac{\Gamma, a, B \to C}{\Gamma, a, a \supset B \to C} \; a \supset L$$

By explicitly writing a in the context, we delay the application of this rule until the point where we could actually apply the usual $\supset L$ and close the left branch.

1.2 Conjunction

When the antecedent is a conjunction, we will use a known logical equivalence to transform the formula into something else. A logical equivalence $A \equiv B$ is defined as $A \supset B$ and $B \supset A$.

$$(A_1 \wedge A_2) \supset B \;\equiv\; A_1 \supset (A_2 \supset B)$$

You might recognize this operation as (un)currying. Intuitively, this will result on A_1 being considered first and, when we need to show A_2, less information will be available. Compare the two derivations below:

$$\dfrac{\dfrac{\Gamma, (A_1 \wedge A_2) \supset B \to A_1 \quad \Gamma, (A_1 \wedge A_2) \supset B \to A_2}{\Gamma, (A_1 \wedge A_2) \supset B \to A_1 \wedge A_2}\wedge R \qquad \Gamma, B \to C}{\Gamma, (A_1 \wedge A_2) \supset B \to C} \supset L$$

$$\dfrac{\Gamma, A_1 \supset (A_2 \supset B) \to A_1 \qquad \dfrac{\Gamma, A_2 \supset B \to A_2 \quad \Gamma, B \to C}{\Gamma, A_2 \supset B \to C}\supset L}{\Gamma, A_1 \supset (A_2 \supset B) \to C} \supset L$$

Therefore, when finding a conjunction as the antecedent of an implication on the left, we will curry it:

$$\dfrac{\Gamma, A_1 \supset (A_2 \supset B) \to C}{\Gamma, (A_1 \wedge A_2) \supset B \to C} \wedge\!\supset L$$

This rule might look strange in the sense that the formula we get in the premise is of the same size as the one in the conclusion. In this case, how do we guarantee termination? After presenting all the rules we will define a measure that gives a greater weight to conjunctions, making the conclusion greater than the premise, and show that in all rules this measure decreases.

1.3 Disjunction

When the antecedent of the implication is a disjunction, we have the following situation:

$$\dfrac{\Gamma, (A_1 \vee A_2) \supset B \to A_1 \vee A_2 \quad \Gamma, B \to C}{\Gamma, (A_1 \vee A_2) \supset B \to C} \supset L$$

If you remember the proof of $\neg\neg(a \vee \neg a)$, this is exactly a case when we might need to use the implication formula on the left again: A_1 might give us something extra to work with and, on the second time, we can choose A_2 (or vice-versa). One intuitive way to avoid the duplication of the implication is to actually split it in two: $A_1 \supset B$ and $A_2 \supset B$.

In this case, instead of choosing A_1 or A_2 with a $\vee R$ rule, we choose the implication to decompose and might be able to get rid of it.

Luckily, we can prove the following logical equivalence:

$$(A_1 \vee A_2) \supset B \;\equiv\; (A_1 \supset B) \wedge (A_2 \supset B)$$

Which we can use to derive the following inference rule:

$$\frac{\Gamma, A_1 \supset B, A_2 \supset B \rightarrow C}{\Gamma, (A_1 \vee A_2) \supset B \rightarrow C} \vee\!\supset L$$

1.4 Implication

When the antecedent of the implication is an implication itself, we have the derivation below. Note that $\supset R$ can be eagerly applied to the left premise because it is an invertible rule.

$$\frac{\dfrac{\Gamma, (A_1 \supset A_2) \supset B, A_1 \rightarrow A_2}{\Gamma, (A_1 \supset A_2) \supset B \rightarrow A_1 \supset A_2} \supset R \qquad \Gamma, B \rightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \rightarrow C} \supset L$$

Observe what happens on the open premise on the left: $(A_1 \supset A_2) \supset B$ and A_1. This means that the antecedent of the red implication depends actually only on A_2, and can thus be simplified. Indeed, we can prove the logical equivalence:

$$((A_1 \supset A_2) \supset B) \wedge A_1 \;\equiv\; (A_2 \supset B) \wedge A_1$$

Which justifies the new inference rule:

$$\frac{\Gamma, A_2 \supset B, A_1 \rightarrow A_2 \quad \Gamma, B \rightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \rightarrow C} \supset\supset L$$

1.5 True

When the antecedent of the implication is $\top$, the transformation is straightforward. The left branch will close and we continue happily with the right branch:

$$\frac{\Gamma, B \rightarrow C}{\Gamma, \top \supset B \rightarrow C} \top \supset L$$

1.6 False

When the antecedent of the implication is $\bot$, we could try to apply $\supset L$ and see how the premises look like, but there is an easier way to simplify the rule. What does $\bot \supset B$ mean? Well, it is the same as $\top$! In fact, we can show the logical equivalence:

$$\bot \supset B \;\equiv\; \top$$

And since we know $\top$ on the left will not make a difference in proof search, we can simply erase it (remember we even had a rule $\top R$ in the inversion calculus which does the same thing):

$$\frac{\Gamma \rightarrow C}{\Gamma, \bot \supset B \rightarrow C} \bot \supset L$$

Remark 1. *In Roy Dyckhoff's referred paper, this system is called LJT, meaning a* ***T****erminating calculus for intuitionistic logic (i.e.* ***LJ****). The name G4ip comes from Troelstra and Schwichtenberg's book,* Basic Proof Theory*. In this book, all sequent calculi are named G, for Gentzen, followed by a number and modifiers. G1i is Gentzen's original sequent calculus system. G2i is one where axioms have the context Γ and weakening, as an explicit rule, is not present. G3i has no explicit contraction rule and finally G4ip is the propositional version of the intuitionistic calculus completely free of contraction.*

Init Rule

$$\frac{}{\Gamma, P \rightarrow P}\ \text{init}$$

Ordinary Rules

$$\frac{}{\Gamma \rightarrow \top}\ \top R \qquad \frac{\Gamma \rightarrow C}{\Gamma, \top \rightarrow C}\ \top L$$

$$\frac{\Gamma \rightarrow A \quad \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}\ \wedge R \qquad \frac{\Gamma, A, B \rightarrow C}{\Gamma, A \wedge B \rightarrow C}\ \wedge L$$

(no $\bot R$ rule)

$$\frac{}{\Gamma, \bot \rightarrow C}\ \bot L$$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow A \vee B}\ \vee R_1 \quad \frac{\Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}\ \vee R_2 \qquad \frac{\Gamma, A \rightarrow C \quad \Gamma, B \rightarrow C}{\Gamma, A \vee B \rightarrow C}\ \vee L$$

$$\frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B}\ \supset R$$

Compound Left Rules

$$\frac{\Gamma, a, B \rightarrow C}{\Gamma, a, a \supset B \rightarrow C}\ P{\supset}L$$

$$\frac{\Gamma, B \rightarrow C}{\Gamma, \top \supset B \rightarrow C}\ \top{\supset}L \qquad \frac{\Gamma, A_1 \supset A_2 \supset B \rightarrow C}{\Gamma, A_1 \wedge A_2 \supset B \rightarrow C}\ \wedge{\supset}L$$

$$\frac{\Gamma \rightarrow C}{\Gamma, \bot \supset B \rightarrow C}\ \bot{\supset}L \qquad \frac{\Gamma, A_1 \supset B, A_2 \supset B \rightarrow C}{\Gamma, A_1 \vee A_2 \supset B \rightarrow C}\ \vee{\supset}L$$

$$\frac{\Gamma, A_2 \supset B, A_1 \rightarrow A_2 \quad \Gamma, B \rightarrow C}{\Gamma, (A_1 \supset A_2) \supset B \rightarrow C}\ {\supset}{\supset}L$$

Figure 1: G4ip: Contraction-free calculus for intuitionistic logic

2 Soundness, Completeness, and Termination

Putting all these rules together gives us the calculus in Figure 1. All the rules in this calculus are invertible, except for $\supset\supset L$ and the usual $\vee R$. So far, we only intuitively argued why these rules are correct. We give more formal arguments now. Henceforth we refer to the simplified sequent calculus as simply sequent calculus.

Theorem 1. *(Soundness) If the sequent $\Gamma \to C$ is derivable in G4ip, then it is derivable in sequent calculus.*

Proof. We prove soundness by structural induction on the proof tree and only show the interesting cases.

BASE CASE: All base cases are trivial because the rules in the contraction-free calculus are the same as in sequent calculus.

INDUCTIVE CASES: All cases are trivial, except the ones for implication left. In all cases below, we assume that $\mathcal{D}'$ and $\mathcal{E}'$ exist by inductive hypothesis.

- The proof ends with $a \supset L$. This is translated directly to a proof in sequent calculus by simply closing the left branch with $init$.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, a, B \to C\end{array}}{\Gamma, a, a \supset B \to C}\ a \supset L \quad\rightsquigarrow\quad \dfrac{\dfrac{}{\Gamma, a, a \supset B \to a}\ init \quad \begin{array}{c}\mathcal{D}'\\ \Gamma, a, B \to C\end{array}}{\Gamma, a, a \supset B \to C}\ \supset L$$

- The proof ends with $\wedge \supset L$. By using the logical equivalence and cut, we can use the IH and get a proof of the same end-sequent in sequent calculus. The left branch can be easily closed and is left as an exercise.

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, A_1 \supset (A_2 \supset B) \to C\end{array}}{\Gamma, (A_1 \wedge A_2) \supset B \to C}\ \wedge \supset L$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, (A_1 \wedge A_2) \supset B, A_1, A_2 \to B}\ \supset L}{\Gamma, (A_1 \wedge A_2) \supset B \to A_1 \supset (A_2 \supset B) \to B}\ \supset R \quad \begin{array}{c}\mathcal{D}' + \text{weak}\\ \Gamma, (A_1 \wedge A_2) \supset B, A_1 \supset (A_2 \supset B) \to C\end{array}}{\Gamma, (A_1 \wedge A_2) \supset B \to C}\ cut$$

- The proof ends with $\vee \supset L$. Again we use cut (this time twice) to derive the end-sequent using the IH. The elided proofs are straightforward.

$$\dfrac{\overset{\mathcal{D}}{\Gamma, A_1 \supset B, A_2 \supset B \to C}}{\Gamma, (A_1 \vee A_2) \supset B \to C} \;\vee\supset L$$

$$\leadsto$$

$$\dfrac{\dfrac{\vdots}{\Gamma, (A_1 \vee A_2) \supset B \to A_1 \supset B} \supset R \qquad \dfrac{\dfrac{\vdots}{\Gamma, (A_1 \vee A_2) \supset B, A_1 \supset B \to A_2 \supset B} \supset R \qquad \overset{\mathcal{D}' + \text{weak}}{\Gamma, (A_1 \vee A_2) \supset B, A_1 \supset B, A_2 \supset B \to C}}{\Gamma, (A_1 \vee A_2) \supset B, A_1 \supset B \to C} \; cut}{\Gamma, (A_1 \vee A_2) \supset B \to C} \; cut$$

- The proof ends with $\supset\supset L$. In this case, we use both implication rules and cut from sequent calculus to obtain a proof of the end-sequent. The trivial proof is elided.

$$\dfrac{\overset{\mathcal{D}}{\Gamma, A_2 \supset B, A_1 \to A_2} \qquad \overset{\mathcal{E}}{\Gamma, B \to C}}{\Gamma, (A_1 \supset A_2) \supset B \to C} \supset\supset L$$

$$\leadsto$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, (A_1 \supset A_2) \supset B, A_1 \to A_2 \supset B} \supset R \qquad \overset{\mathcal{D}' + \text{weak}}{\Gamma, (A_1 \supset A_2) \supset B, A_2 \supset B, A_1 \to A_2}}{\Gamma, (A_1 \supset A_2) \supset B, A_1 \to A_2} \; cut}{\Gamma, (A_1 \supset A_2) \supset B \to A_1 \supset A_2} \supset R \qquad \overset{\mathcal{E}'}{\Gamma, B \to C}}{\Gamma, (A_1 \supset A_2) \supset B \to C} \supset L$$

- The proof ends with $\top \supset B$. This is translated directly into a sequent calculus proof which applies $\supset L$ to the formula and closes the left branch with $\top R$, much like the atomic case.

- The proof ends with $\bot \supset B$. The soundness of this rule is immediate due to admissibility of weakening.

□

Completeness, as usual, is more complicated. Its proof, as well as the proof of termination, relies on a measure of *weight* of a sequent. We will use the measure defined by Troelstra and Schwichtenberg in [2].

Definition 1. *(Weight) For each propositional formula A, we assing it a* weight *as follows:*

- $w(A) = w(\bot) = w(\top) = 2$ *for atomic A.*
- $w(A \wedge B) = w(A)(1 + w(B))$
- $w(A \vee B) = 1 + w(A) + w(B)$
- $w(A \supset B) = 1 + w(A)w(B)$

They weight of a sequent $\Gamma \to C$ is thus defined as:

$$w(\Gamma \to C) = \sum \left\{ w(B) \mid B \in \Gamma \cup \{C\} \right\}$$

Notice that the weights are natural numbers starting from 2.

Lemma 1. *For every rule in G4ip, its premises have a strictly lower weight than its conclusion.*

Proof. The lemma holds by inspection of each rule. We only show the case for $\wedge \supset L$, which is the less intuitive one. The formulas in Γ and C are unchanged, so their weight is the same on both conclusion and premise. We analyse thus the weights of the main and auxiliary formulas:

$$\begin{aligned} w(A_1 \supset (A_2 \supset B)) &= 1 + w(A_1)(w(A_2 \supset B)) \\ &= 1 + w(A_1)(1 + w(A_2)w(B)) \\ &= 1 + w(A_1) + w(A_1)w(A_2)w(B) \end{aligned}$$

$$\begin{aligned} w((A_1 \wedge A_2) \supset B) &= 1 + w(A_1 \wedge A_2)w(B) \\ &= 1 + (w(A_1)(1 + w(A_2)))w(B) \\ &= 1 + (w(A_1) + w(A_1)w(A_2))w(B) \\ &= 1 + w(A_1)w(B) + w(A_1)w(A_2)w(B) \end{aligned}$$

The only difference is on the second term: $w(A_1)$ for the premise and $w(A_1)w(B)$ for the conclusion. Since the smallest weight a formula can have is 2 (atomic case), it is definitely the case that $w(A_1) < w(A_1)w(B)$, and thus the weight of the premise is smaller than the weight of the conclusion.

□

Theorem 2. *(Termination) Proof search in G4ip is terminating.*

Proof. Suppose we are looking for a proof of a sequent $\Gamma \to C$. From Lemma 1 we know that every rule's premises are strictly smaller than its conclusion. So a proof of $\Gamma \to C$ can have at most $w(\Gamma \to C)$ steps. Searching for a proof with this bound on its height is a terminating procedure to decide if the sequent is provable or not. □

Although we have a decision procedure for intuitionistic logic with this calculus, the order in which we apply the rules still might make a difference. So we may need to backtrack during proof search. We can use the ideas from the inversion calculus to improve proof search on this calculus as well.

Theorem 3. *(Completeness) If a sequent $\Gamma \to C$ is provable in sequent calculus, then it is also provable in G4ip.*

Proof. Contrary to what you may think, this proof follows by **induction on the weight of the sequent**.

BASE CASE: The smallest possible provable sequent is one with only two atoms, of size 4: $a \to a$. This is certainly provable in G4ip.

INDUCTIVE CASES: For the inductive cases, we consider what might be the formulas in the sequent $\Gamma \to C$, assuming it has a sequent calculus proof.

1. $\Gamma \;\equiv\; \Gamma', \bot$

   This sequent is trivially provable in G4ip by an application of $\bot L$.

2. $\Gamma \;\equiv\; \Gamma', A \wedge B$

   By invertibility of $\wedge L$, there exists a sequent calculus proof of $\Gamma', A, B \to C$. Since this is a smaller sequent, it also has a proof in G4ip by the IH and thus we can use $\wedge L$ in G4ip to obtain a proof of $\Gamma', A \wedge B \to C$.

3. $\Gamma \;\equiv\; \Gamma', A \vee B$

   Again, by invertibility of $\vee L$, there are sequent calculus proofs of $\Gamma', A \to C$ and $\Gamma', B \to C$. Since both these sequents are smaller than $\Gamma', A \vee B \to C$, we can apply the IH and get G4ip proofs of them. By an application of $\vee L$ we get a proof of the same end-sequent in G4ip.

4. $\Gamma \;\equiv\; \Gamma', a, a \supset B$

   From a derivation of $\Gamma', a, a \supset B \to C$ we can obtain a derivation of $\Gamma', a, B \to C$, as demonstrated by the transformation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, a, B, a \to B}\, id}{\Gamma, a, B \to a \supset B}\supset R \qquad \Gamma', a, a \supset B \to C}{\Gamma', a, B \to C}\, cut$$

By the IH, there exists a G4ip proof of $\Gamma', a, B \to C$, to which we can apply rule $a \supset L$ and get a proof of $\Gamma', a, a \supset B \to C$.

5. For the other cases, we analyse what is the lower most rule in the proof of $\Gamma \to C$. We will use the following Lemma:

   Lemma 2. *If a sequent $\Gamma \to C$ is such that Γ does not contain $\bot$, conjunctions, disjunctions or a pair $a, a \supset B$, then it has a proof ending with either an application of a right rule or a left implication on $A \supset B$ where A is* not *an atom.*

   If a sequent does not have any of the mentioned formulas, it only contains atoms and implications on the left side. So, in principle, for this case in the completeness proof we would have to consider proofs that end with the following rules: $\wedge R$, $\vee R$, $\supset R$ and $\supset L$. What this lemma tells us is that we can restrict the $\supset L$ case to formulas $A \supset B$ where A is not atomic.

   The right cases are straightforward.

   (a) The sequent calculus proof of $\Gamma \to C$ ends with $\wedge R$, $\vee R$ or $\supset R$.

   In all those cases, the premises are smaller sequents than the conclusion, so we can use the IH to argue that G4ip proofs exist. Using the same corresponding rules we obtain proofs of the same end-sequent in G4ip.

   Observe that the cases of $\wedge R$ and $\supset R$ could have been proved with the same invertibility argument as the left invertible rules.

   (b) The sequent calculus proof of $\Gamma \to C$ ends with $\supset L$ on a formula $A \supset B$.

   Since we know from the above Lemma that A cannot be atomic, we case on its possible formats.

   i. A is a conjunction $A_1 \wedge A_2$.
   From a sequent calculus proof of $\Gamma', (A_1 \wedge A_2) \supset B \to C$ we can obtain a proof of $\Gamma', A_1 \supset (A_2 \supset B) \to C$ in the following way:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma', A_1 \supset (A_2 \supset B), A_1, A_2 \to B} \supset L}{\Gamma', A_1 \supset (A_2 \supset B) \to (A_1 \wedge A_2) \supset B} \supset R, \wedge L \qquad \Gamma', (A_1 \wedge A_2) \supset B \to C}{\Gamma', A_1 \supset (A_2 \supset B) \to C} \; cut$$

   Since $\Gamma', A_1 \supset (A_2 \supset B) \to C$ is smaller than $\Gamma', (A_1 \wedge A_2) \supset B \to C$, we can apply the IH and get a G4ip proof of it. By applying the rule $\wedge \supset L$ we obtain a proof of the end-sequent we want.

ii. A is a disjunction $A_1 \vee A_2$.
Similar to the previous case, we can construct the proof:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma', A_1 \supset B, A_2 \supset B, A_1 \vee A_2 \rightarrow B}}{\Gamma', A_1 \supset B, A_2 \supset B \rightarrow (A_1 \vee A_2) \supset B}\supset R \qquad \Gamma', (A_1 \vee A_2) \supset B \rightarrow C}{\Gamma', A_1 \supset B, A_2 \supset B \rightarrow C}\mathit{cut}$$

Since $\Gamma', A_1 \supset B, A_2 \supset B \rightarrow C$ is a smaller sequent, we can apply the IH and get a G4ip proof of it. By applying $\vee \supset L$ we get a G4ip proof of $\Gamma', (A_1 \vee A_2) \supset B \rightarrow C$, as desired.

iii. A is an implication $A_1 \supset A_2$.
Then we have sequent calculus proofs of the following premises:

$$\Gamma', (A_1 \supset A_2) \supset B \rightarrow A_1 \supset A_2 \qquad \Gamma', B \rightarrow C$$

By the IH, we have a G4ip proof of $\Gamma', B \rightarrow C$.
By invertibility of $\supset R$, we have a sequent calculus proof of $\Gamma', (A_1 \supset A_2) \supset B, A_1 \rightarrow A_2$. Using this proof, we can obtain a (sequent calculus) proof of $\Gamma', A_2 \supset B, A_1 \rightarrow A_2$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma', A_2 \supset B, A_1, A_1 \supset A_2 \rightarrow B}}{\Gamma', A_2 \supset B, A_1 \rightarrow (A_1 \supset A_2) \supset B}\supset R \qquad \Gamma', (A_1 \supset A_2) \supset B, A_1 \rightarrow A_2}{\Gamma', A_2 \supset B, A_1 \rightarrow A_2}\mathit{cut}$$

Now this sequent is smaller and we can apply the IH to get a G4ip proof of it. Using this and the previous proof obtained by the IH, we apply $\supset\supset L$ and get a G4ip proof of the desired end-sequent.

iv. A is $\bot$.
Using the sequent calculus proof of $\Gamma', \bot \supset B \rightarrow C$ we get a proof of $\Gamma' \rightarrow C$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma', \bot \rightarrow C}\bot L}{\Gamma' \rightarrow \bot \supset B}\supset R \qquad \Gamma', \bot \supset B \rightarrow C}{\Gamma' \rightarrow C}\mathit{cut}$$

Since this sequent is smaller, we use the IH to state that there exists a G4ip proof of it. Using $\bot \supset L$ we obtain a proof of the desired end-sequent.

□

References

[1] Roy Dyckhoff. Contraction-Free Sequent Calculi for Intuitionistic Logic. *J. Symb. Log.*, 57(3):795–807, 1992.

[2] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2 edition, 2000.