Compressing Polarized Boxes

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Proof nets: the graphical syntax for linear logic.

Brought new deep perspectives about normalization:

1. Optimal reductions;
2. Implicit computational complexity;
3. Explicit substitutions;
4. Strong normalization.

Key tool: boxes for the promotion rule, the heart of the system.

This work: a new understanding of boxes, via polarity.
### Multiplicative Linear Logic (MLL)

**Identity rules:**

$$\vdash A\bot, A$$  \text{ax} \\
$$\vdash \Gamma, A\bot, \Delta$$  \text{cut}

**Multiplicative rules:**

$$\vdash \Gamma, \Delta, A \otimes B$$  \otimes \\
$$\vdash \Gamma, A \bowtie B$$

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Compressing Polarized Boxes
Proof nets for MLL

\[ \vdash A \bot, A \quad \text{ax} \]

\[ \vdash \Gamma, A \quad \vdash \Delta, A \bot \quad \text{cut} \]

\[ \vdash \Gamma, \Delta \quad \text{cut} \]

\[ \vdash \Gamma, A, B \quad \text{cut} \]

\[ \vdash \Gamma, A \otimes B \quad \otimes \]

\[ \vdash \Gamma, \Delta, A \otimes B \quad \otimes \]

\[ \vdash \Gamma, A \otimes B \quad \otimes \]

\[ \vdash \Gamma, \Delta, A \otimes B \quad \otimes \]
Cut-elimination for MLL

No duplication/erasure of subnets

⇒

Everything works fine
Multiplicative Exponential Linear Logic (MELL)

**MLL**

+ 

**Exponential rules:**

\[
\begin{align*}
\Gamma, A & \vdash \Gamma, ?A & \Gamma, ?A & \vdash \Gamma, A \\
\Gamma, ?A & \vdash \Gamma, ?A & \Gamma, ?A & \vdash \Gamma, ?A
\end{align*}
\]

\[
\begin{align*}
?\Gamma, A & \vdash \Gamma, ?A & ?\Gamma, A & \vdash \Gamma, !A \\
?\Gamma, !A & \vdash ?\Gamma, A & ?\Gamma, A & \vdash ?\Gamma, A
\end{align*}
\]
Exponential Cut-elimination

Consider the following cut with contraction:

\[
\begin{array}{c}
\rho \\
\vdash \neg \Delta, A \\
\vdash \neg \Delta, !A \\
\vdash \neg \Delta, \Gamma \\
\end{array}
\]

Its elimination requires to duplicate \(\rho\):

\[
\begin{array}{c}
\rho \\
\vdash \neg \Delta, A \\
\vdash \neg \Delta, !A \\
\vdash \neg \Delta, \Gamma \\
\end{array}
\]

Similarly, weakening induces erasure of sub-proofs.
Naïve proof nets for MELL

\[
\pi : \Gamma \vdash \Gamma, \ ?A \\
\vdash \Gamma, \ ?A, \ A \\
\vdash \Gamma, \ ?A
\]

\[
\pi : \Gamma, \ A \vdash \Gamma, \ ?A \\
\vdash \Gamma, \ ?A
\]

\[
\pi : \Gamma, \ ?A, \ ?A \vdash \Gamma, \ ?A \\
\vdash \Gamma, \ ?A
\]

\[
\pi : \ ?\Gamma, \ A \vdash \ ?\Gamma, \ !A \\
\vdash \ ?\Gamma, \ !A
\]

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How to eliminate cuts?

Naïve translation of promotion:

$$
\frac{\pi}{\vdash ?\Gamma, A} \quad \vdash ?\Gamma, !A
\quad \frac{\pi^*}{!A}
\quad \vdash ?\Gamma, !A
$$

Given this cut in a generic net:

There is no way of recovering the sub-proof to duplicate.

Then !-rules are represented as boxes:
Exponential cut elimination implemented using boxes

\[ \begin{array}{c}
\text{w} \\
\text{?A⊥} \\
\text{cut} \\
\text{!A} \\
\text{?B_1} \\
\text{?B_k} \\
\rightarrow \text{w} \\
\text{?B_1} \\
\text{?B_k} \\
\text{w} \\
\end{array} \]

\[ \begin{array}{c}
\text{d} \\
\text{A⊥} \\
\text{cut} \\
\text{A} \\
\text{?A⊥} \\
\rightarrow \text{d} \\
\text{A} \\
\text{?A⊥} \\
\text{Γ} \\
\end{array} \]

\[ \begin{array}{c}
\text{c} \\
\text{?A⊥} \\
\text{?A⊥} \\
\text{c} \\
\text{?A⊥} \\
\text{cut} \\
\text{!A} \\
\text{?B_1} \\
\text{?B_k} \\
\rightarrow \text{c} \\
\text{?B_1} \\
\text{?B_k} \\
\text{?A⊥} \\
\text{Γ} \\
\end{array} \]

\[ \begin{array}{c}
\text{□} \\
\text{!} \\
\text{A} \\
\text{?Δ} \\
\text{cut} \\
\text{?Γ} \\
\rightarrow \text{□} \\
\text{!} \\
\text{B} \\
\text{?Γ} \\
\end{array} \]

\[ \begin{array}{c}
\text{□} \\
\text{!} \\
\text{P} \\
\text{!} \\
\text{A} \\
\text{?Δ} \\
\text{cut} \\
\text{?Γ} \\
\rightarrow \text{□} \\
\text{!} \\
\text{B} \\
\text{?Γ} \\
\end{array} \]

\[ \begin{array}{c}
\text{□} \\
\text{!} \\
\text{P} \\
\text{!} \\
\text{A} \\
\text{?Δ} \\
\text{cut} \\
\text{?Γ} \\
\rightarrow \text{□} \\
\text{!} \\
\text{B} \\
\text{?Γ} \\
\end{array} \]
• Boxes **solve the problem** of defining cut-elimination.

• However, the solution is **drastic**, equivalent to **give up**.

• Some fragments seem to have an **inherent notion of box**.

• Where does the problem lie?

• Is there a **logic feature** that **internalizes boxes**?
Main problem: in proof nets there is no last rule.

Re-consider:

\[
\begin{align*}
\rho & : \vdash \Delta, A \\
\pi & : \\
\vdash \Delta, !A & ! \\
\vdash \Delta, \perp & , \Delta, \Gamma & \text{cut} \\
\vdash \Delta, \Gamma & \\
\rightarrow & \\
\rho & : \vdash \Delta, A \\
\vdash \Delta, !A & ! \\
\vdash \Delta, \perp & , \Delta, \Gamma & \text{cut} \\
\vdash \Delta, \perp & , \Delta, \Gamma & \text{cut} \\
\vdash \Delta, \Gamma & \\
\vdash \Delta, \Gamma & \\
\end{align*}
\]

In sequent calculus:

rule occurrence \( r \mapsto \text{sub-proof} \) ending on \( r \).

No such thing in proof nets!
• **Intuition:**

  Internalizing a notion of **last rule**
  will internalize **boxes**

• **Partially internalized boxes:** Olivier Laurent’s **polarized MELL**.

• **Abstract last rule** = **last positive rule**.

• **This work:** **totally internalized boxes** for **MELLP**.

• **Expressiveness:** **MELLP** codes **classical logic/λμ-calculus**.
Outline

1. Polarized MELL

2. Compressing polarized boxes
Polarization

Formulas:

\[
P, Q ::= X \mid 1 \mid P \otimes Q \mid !N
\]

\[
N, M ::= X^\perp \mid \bot \mid N \otimes M \mid ?P
\]

Sequents:

\[
\vdash \Gamma ; P \quad \text{or} \quad \vdash \Gamma ; \bot
\]

Multiplicative rules:

\[
\begin{array}{c}
\vdash \Gamma; P \\
\vdash \Delta, P^\perp; [Q]
\end{array}
\]

\[
\frac{\vdash \Gamma; P \quad \vdash \Delta, P^\perp; [Q]}{\vdash \Gamma, \Delta; [Q]} \quad \text{cut}
\]

\[
\begin{array}{c}
\vdash \Gamma; [P] \\
\vdash \Gamma, \bot; [P]
\end{array}
\]

\[
\frac{\vdash \Gamma; [P]}{\vdash \Gamma, \bot; [P]} \quad \bot
\]

\[
\frac{\vdash \Gamma; N, M; [P]}{\vdash \Gamma, N \otimes M; [P]} \quad \otimes
\]

\[
\frac{\vdash \Gamma; P \quad \vdash \Delta; Q}{\vdash \Gamma, \Delta; P \otimes Q} \quad \otimes
\]
Laurent’s MELLP: adding exponentials

- Exponential rules:

\[
\frac{\Gamma; [P]}{\Gamma, N; [P]} \quad w \quad \frac{\Gamma; P}{\Gamma, ?P; -} \quad d
\]

\[
\frac{\Gamma, N, N; [P]}{\Gamma, N; [P]} \quad c \quad \frac{\Gamma, N; -}{\Gamma; !N} \quad !
\]

- Difference with linear logic:

Promotion, contraction, and weakening do not need the \( ? \) modality.

- Important:

Only \textbf{positives} are duplicated/erased.

- Positives are last rules and every positive will have a \textbf{box}.
\[\vdash 1 \quad \leadsto \quad 1 \uparrow\]

\[\vdash P \perp; P \quad \leadsto \quad \vdash P \perp \quad \vdash P \]

\[\vdash \Gamma; P \quad \vdash \Delta; Q \quad \vdash \Gamma, \Delta; P \otimes Q \quad \otimes \quad \leadsto \quad \vdash \Gamma \quad \pi^* \quad \vdash \Delta \quad \theta^* \quad \vdash P \otimes Q \quad \Gamma \quad \Delta \]

\[\vdash \Gamma; P \quad \vdash \Delta, P \perp; [Q] \quad \text{cut} \quad \leadsto \quad \vdash \Gamma, \Delta; [Q] \quad \vdash \Gamma \quad \pi^* \quad \vdash P \quad \text{cut} \quad \vdash P \perp \quad \theta^* \quad \vdash \Delta; [Q] \quad \Gamma \quad \Delta \]

\[\vdash \Gamma, N, M; [P] \quad \leadsto \quad \vdash \Gamma, N \not\supset M; [P] \quad \vdash \Gamma \quad \pi^* \quad \vdash N \quad \not\supset \quad \vdash M \quad [P] \quad N \not\supset \quad M \]
\[ \frac{\vdash \Gamma; [P]}{\vdash \Gamma, N; [P]} \quad w \quad \rightsquigarrow \quad \vdash \Gamma, N; [P] \]

\[ \frac{\vdash \Gamma; P}{\vdash \Gamma, ?P; \_} \quad d \quad \rightsquigarrow \quad \vdash \Gamma, ?P; \_ \]

\[ \frac{\vdash \Gamma, N, N; [P]}{\vdash \Gamma, N; [P]} \quad c \quad \rightsquigarrow \quad \vdash \Gamma, N; [P] \]

\[ \frac{\vdash \Gamma, N; \_}{\vdash \Gamma; !N} \quad ! \quad \rightsquigarrow \quad \vdash \Gamma; !N \]
Positive Trees

\[
\begin{align*}
\frac{\Gamma; P \quad \Delta, P^\perp; [Q]}{\Gamma, \Delta; [Q]} & \quad \text{cut} \\
\frac{\Gamma; [P]}{\Gamma, \bot; [P]} & \quad \bot \\
\frac{\Gamma, N, M; [P]}{\Gamma, N \otimes M; [P]} & \quad \otimes \\
\frac{\Gamma; [P]}{\Gamma, N; [P]} & \quad w \\
\frac{\Gamma, N, N; [P]}{\Gamma, N; [P]} & \quad c \\
\frac{\Gamma; P}{\Gamma; P^\perp; P} & \quad \text{ax} \\
\frac{\Gamma; P}{\Gamma, \Delta; P \otimes Q} & \quad \otimes \\
\frac{\Gamma; P}{\Gamma, \bot; [P], \bot} & \quad \bot \\
\frac{\Gamma, N; [P]}{\Gamma, N; [P]} & \quad w \\
\frac{\Gamma, N; -}{\Gamma, !N} & \quad !
\end{align*}
\]

Note: positives have a forest structure.
Positive Tree

- **Positive** connectives: $1, \otimes, !$.
- **Explicit** boxes for $! \Rightarrow$ **induced box** for every positive:

  \[
  \begin{array}{c}
  H \\
  ! \\
  N_1 \ldots N_k \\
  \end{array} \quad \rightarrow \quad \begin{array}{c}
  H \\
  ! \\
  N_1 \ldots N_k \\
  \end{array} \quad \rightarrow \quad \begin{array}{c}
  \text{ax} \quad P \quad P_\perp \\
  \end{array} \quad \rightarrow \quad \begin{array}{c}
  \text{ax} \quad P \\
  \end{array}
  \]

  \[
  \begin{array}{c}
  P \otimes Q \\
  P_\perp \\
  \end{array} \quad \rightarrow \quad \begin{array}{c}
  P \otimes Q \\
  P_\perp \\
  \end{array}
  \]

- **My contribution**: **explicit** boxes for $!$ can be made **implicit**.
Laurent uses the **positive tree** to generalize box rules:

![Diagram](image)
Polarized cut-elimination 1

\[
\begin{array}{c}
P \xrightarrow{\text{ax}} \ p \xrightarrow{\perp} \ p \xrightarrow{\text{cut}} \ ax^- \\
N \xrightarrow{\text{ax}} \ n \xrightarrow{\perp} \ n \xrightarrow{\text{cut}} \ ax^+ \\
\end{array}
\]
Polarized cut-elimination 2

\[
E \rightarrow_d d
\]

\[
T(+) \rightarrow_w w
\]

\[
P \perp P \perp \rightarrow_c c
\]

\[
Q \rightarrow_h h
\]
Outline

1. Polarized MELL

2. Compressing polarized boxes
Matching property: every !-rule is enabled by a d-rule.
Materializing the matching property

- Consider:

  ![Diagram](image)

- **Problem**: without box the content is disconnected.

- **Idea**: let’s materialize the matching property with an additional edge.

  ![Diagram](image)

- The content and the positive sub-graphs are now **connected**.

- **The induced box**: the **positive tree** plus the **negative trees** on it.
Let’s do it again:

We do not recover the original box:

**Interpretation**: we are quotienting proof nets with explicit boxes.

**Remark**: weakenings are not attached!

⇒ improvement over Francois Lamarche’s essential nets.
Let's do it again:

Remark: we are not attaching the border of the box.

⇒ improvement over Ian Mackie's interaction nets technique.
Implicit boxes

Recipe:

- Take a \textbf{cut-free proof net}.
- **Matching**: every !-box has a \textbf{unique dereliction at level 0}.
- Remove the explicit box and add the matching edge.

Then:

- The \textbf{induced boxes} define a net with explicit boxes.
- Induced boxes are \textbf{locally reconstructable}.
- There is a simple \textbf{correctness criterion} (i.e. not \textit{ad-hoc}).
- It is a \textbf{canonical} representation (i.e. no choice).
In a cut-free proof net

the explicit box of a !

can be replaced by a single edge

in a canonical and more parallel way.
Cuts introduce a **problem**:

The **positive sub-graph** is **no longer connected**.

Let’s iterate **the same idea**:
Example

- **Implicit box**: one dereliction plus the cuts at level 0.
- **Induced box**: positive tree plus negative sub-trees.
- **Novelty**: \( \mathcal{R} \) commutes with box borders!
Cut elimination
Cut elimination has 'side effects'.

Consider the weakening rule:

It automatically pushes the created weakenings out of boxes:

Similarly for contraction.
There is no commutative rule. It is included inside the axiom and dereliction rules.

The axiom rule:

\[
\begin{align*}
X & \xrightarrow{\text{ax}} X \\
& \xrightarrow{\text{cut}} ! \\
& \xrightarrow{\text{ax}} X \\
& \xrightarrow{\text{!}} i
\end{align*}
\]

and its action through box borders:

\[
\begin{align*}
P & \xrightarrow{\text{ax}} \ldots \\
& \xrightarrow{\text{cut}} ! \\
& \xrightarrow{\text{!}} i \\
& \xrightarrow{\text{ax}} \sqotient(+) \\
& \xrightarrow{\text{!}} i \\
& \xrightarrow{\text{ax}} \sqotient(+) \\
& \xrightarrow{\text{!}} i
\end{align*}
\]
The dereliction rule:

![Diagram of dereliction rule]

and an example of its action:

![Diagram of example action]
Cut-elimination

On $\eta$-expanded nets:

- Cut elimination is **strongly normalizing** (SN).
- Confluence requires **assoc.**, **comm.**, and **neutrality** for contractions.
- Then: **Church-Rosser modulo** and **SN modulo** hold.

The general case:

- More and wilder ’**side effects**’.
- **Difficult critical pairs** and known techniques do **not work**.
- **By-product**: new simpler proof of SN for linear logic (RTA 2013).
Conclusions

- An **alternative representation of boxes**: 
  - Simple;
  - Canonical;
  - More parallel;
  - Provided of a correctness criterion;
  - A local reconstruction of boxes;
  - Results on the dynamics.

- **New perspective on polarity**.

- It already lead to **new understanding of SN** for linear logic.
THANKS!