9.1 A Chaotic Waterwheel

A neat mechanical model of the Lorenz equations was invented by Willem Malkus and Lou Howard at MIT in the 1970s. The simplest version is a toy waterwheel with leaky paper cups suspended from its rim (Figure 9.1.1).

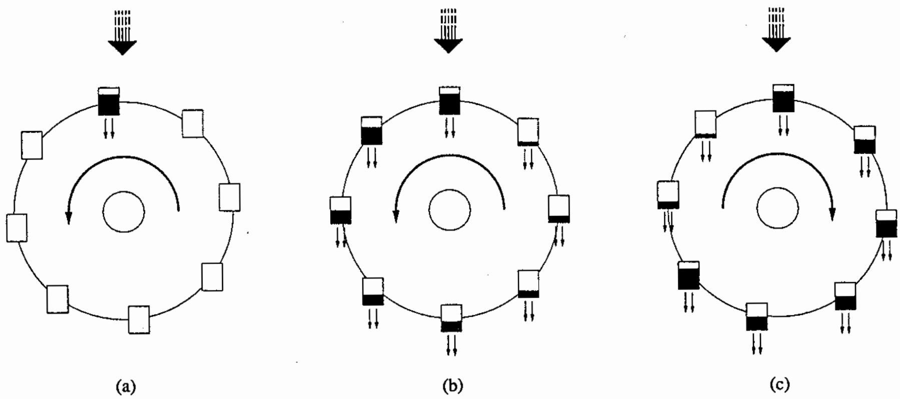

Figure 9.1.1

Water is poured in steadily from the top. If the flow rate is too slow, the top cups never fill up enough to overcome friction, so the wheel remains motionless. For faster inflow, the top cup gets heavy enough to start the wheel turning (Figure 9.1.1a). Eventually the wheel settles into a steady rotation in one direction or the other (Figure 9.1.1b). By symmetry, rotation in either direction is equally possible; the outcome depends on the initial conditions.

By increasing the flow rate still further, we can destabilize the steady rotation. Then the motion becomes chaotic: the wheel rotates one way for a few turns, then some of the cups get too full and the wheel doesn't have enough inertia to carry them over the top, so the wheel slows down and may even reverse its direction (Figure 9.1.1c). Then it spins the other way for a while. The wheel keeps changing direction erratically. Spectators have been known to place bets (small ones, of course) on which way it will be turning after a minute.

Figure 9.1.2 shows Malkus's more sophisticated set-up that we use nowadays at MIT.

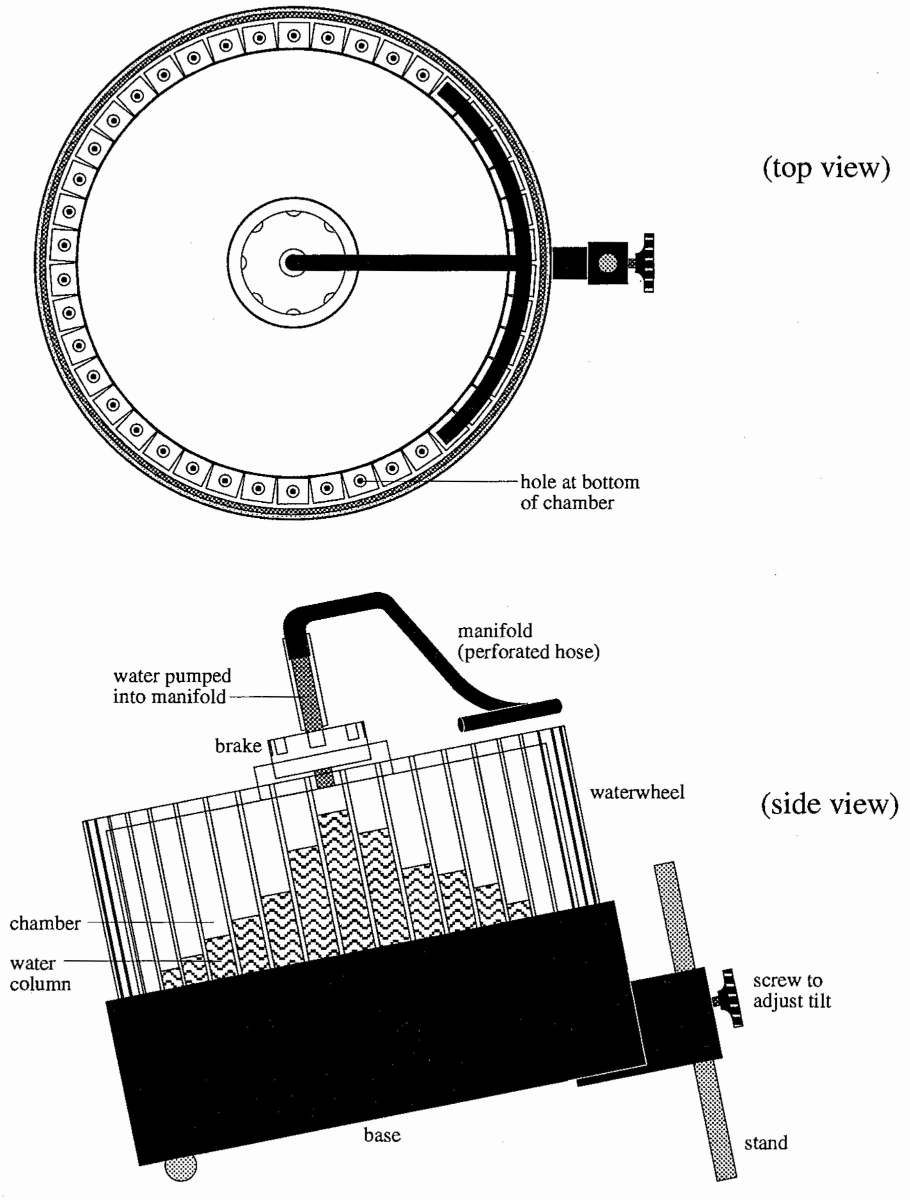

Figure 9.1.2

The wheel sits on a table top. It rotates in a plane that is tilted slightly from the horizontal (unlike an ordinary waterwheel, which rotates in a vertical plane). Water is pumped up into an overhanging manifold and then sprayed out through dozens of small nozzles. The nozzles direct the water into separate chambers around the rim of the wheel. The chambers are transparent, and the water has food coloring in it, so the distribution of water around the rim is easy to see. The water leaks out through a small hole at the bottom of each chamber, and then collects underneath the wheel, where it is pumped back up through the nozzles. This system provides a steady input of water.

The parameters can be changed in two ways. A brake on the wheel can be adjusted to add more or less friction. The tilt of the wheel can be varied by turning a screw that props the wheel up; this alters the effective strength of gravity.

A sensor measures the wheel's angular velocity $\omega(t)$, and sends the data to a strip chart recorder which then plots $\omega(t)$ in real time. Figure 9.1.3 shows a record of $\omega(t)$ when the wheel is rotating chaotically. Notice once again the irregular sequence of reversals.

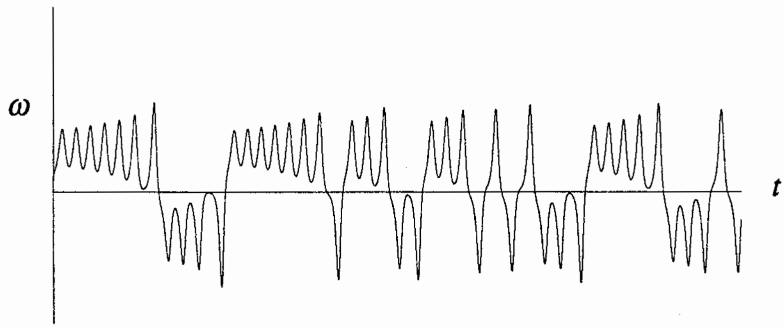

Figure 9.1.3

We want to explain where this chaos comes from, and to understand the bifurcations that cause the wheel to go from static equilibrium to steady rotation to irregular reversals.

Notation

Here are the coordinates, variables and parameters that describe the wheel's motion (Figure 9.1.4):

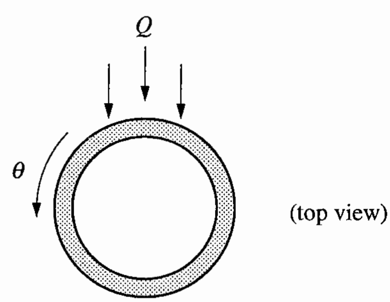

Figure 9.1.4

θ = angle in the lab frame (*not* the frame attached to the wheel)
$\theta = 0 \leftrightarrow$ 12:00 in the lab frame

$\omega(t)$ = angular velocity of the wheel (increases counterclockwise, as does θ)
$m(\theta,t)$ = mass distribution of water around the rim of the wheel, defined such that the mass between θ_1 and θ_2 is $M(t) = \int_{\theta_1}^{\theta_2} m(\theta,t)\, d\theta$
$Q(\theta)$ = inflow (rate at which water is pumped in by the nozzles above position θ)
r = radius of the wheel
K = leakage rate
ν = rotational damping rate
I = moment of inertia of the wheel

The unknowns are $m(\theta,t)$ and $\omega(t)$. Our first task is to derive equations governing their evolution.

Conservation of Mass

To find the equation for conservation of mass, we use a standard argument. You may have encountered it if you've studied fluids, electrostatics, or chemical engineering. Consider any sector $[\theta_1, \theta_2]$ fixed in space (Figure 9.1.5).

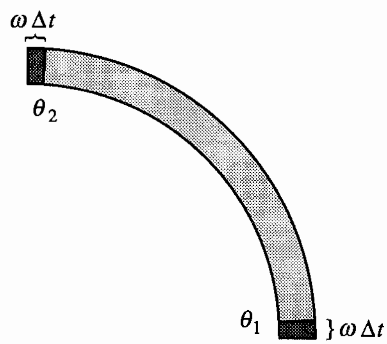

Figure 9.1.5

The mass in that sector is $M(t) = \int_{\theta_1}^{\theta_2} m(\theta,t)\, d\theta$. After an infinitesimal time Δt, what is the change in mass ΔM? There are four contributions:

1. The mass pumped in by the nozzles is $\left[\int_{\theta_1}^{\theta_2} Q d\theta\right] \Delta t$.
2. The mass that leaks out is $\left[-\int_{\theta_1}^{\theta_2} Km\; d\theta\right] \Delta t$. Notice the factor of m in the integral; it implies that leakage occurs at a rate proportional to the mass of water in the chamber—more water implies a larger pressure head and therefore faster leakage. Although this is plausible physically, the fluid mechanics of leakage is complicated, and other rules are conceivable as

well. The real justification for the rule above is that it agrees with direct measurements on the waterwheel itself, to a good approximation. (For experts on fluids: to achieve this linear relation between outflow and pressure head, Malkus attached thin tubes to the holes at the bottom of each chamber. Then the outflow is essentially Poiseuille flow in a pipe.)

3. As the wheel rotates, it carries a new block of water into our observation sector. That block has mass $m(\theta_1)\omega\Delta t$, because it has angular width $\omega\Delta t$ (Figure 9.1.5), and $m(\theta_1)$ is its mass per unit angle.
4. Similarly, the mass carried out of the sector is $-m(\theta_2)\omega\Delta t$.

Hence,

$$\Delta M = \Delta t\left[\int_{\theta_1}^{\theta_2} Q\,d\theta - \int_{\theta_1}^{\theta_2} Km\,d\theta\right] + m(\theta_1)\omega\Delta t - m(\theta_2)\omega\Delta t. \qquad (1)$$

To convert (1) to a differential equation, we put the transport terms inside the integral, using $m(\theta_1)-m(\theta_2)=-\int_{\theta_1}^{\theta_2}\frac{\partial m}{\partial\theta}\,d\theta$. Then we divide by Δt and let $\Delta t \to 0$. The result is

$$\frac{dM}{dt} = \int_{\theta_1}^{\theta_2}\left(Q - Km - \omega\tfrac{\partial m}{\partial\theta}\right)d\theta.$$

But by definition of M,

$$\frac{dM}{dt} = \int_{\theta_1}^{\theta_2}\frac{\partial m}{\partial t}\,d\theta.$$

Hence

$$\int_{\theta_1}^{\theta_2}\frac{\partial m}{\partial t}\,d\theta = \int_{\theta_1}^{\theta_2}\left(Q - Km - \omega\tfrac{\partial m}{\partial\theta}\right)d\theta.$$

Since this holds for *all* θ_1 and θ_2, we must have

$$\frac{\partial m}{\partial t} = Q - Km - \omega\frac{\partial m}{\partial\theta}. \qquad (2)$$

Equation (2) is often called the ***continuity equation***. Notice that it is a *partial* differential equation, unlike all the others considered so far in this book. We'll worry about how to analyze it later; we still need an equation that tells us how $\omega(t)$ evolves.

Torque Balance

The rotation of the wheel is governed by Newton's law $F = ma$, expressed as a balance between the applied torques and the rate of change of angular momentum. Let I denote the moment of inertia of the wheel. Note that in general I depends on

t, because the distribution of water does. But this complication disappears if we wait long enough: as $t \to \infty$, one can show that $I(t) \to$ constant (Exercise 9.1.1). Hence, after the transients decay, the equation of motion is

$$I\dot{\omega} = \text{damping torque} + \text{gravitational torque}.$$

There are two sources of damping: viscous damping due to the heavy oil in the brake, and a more subtle "inertial" damping caused by a spin-up effect—the water enters the wheel at zero angular velocity but is spun up to angular velocity ω before it leaks out. Both of these effects produce torques proportional to ω, so we have

$$\text{damping torque} = -\nu\omega,$$

where $\nu > 0$. The negative sign means that the damping opposes the motion.

The gravitational torque is like that of an inverted pendulum, since water is pumped in at the top of wheel (Figure 9.1.6).

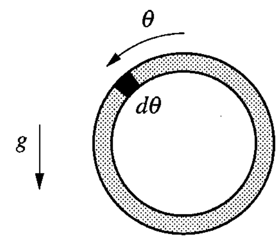

Figure 9.1.6

In an infinitesimal sector $d\theta$, the mass $dM = m d\theta$. This mass element produces a torque

$$d\tau = (dM) gr \sin\theta = mgr \sin\theta \, d\theta.$$

To check that the sign is correct, observe that when $\sin\theta > 0$ the torque tends to *increase* ω, just as in an inverted pendulum. Here g is the effective gravitational constant, given by $g = g_0 \sin\alpha$ where g_0 is the usual gravitational constant and α is the tilt of the wheel from horizontal (Figure 9.1.7).

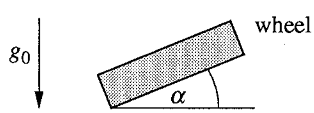

Figure 9.1.7

Integration over all mass elements yields

$$\text{gravitational torque} = gr \int_0^{2\pi} m(\theta,t) \sin\theta \, d\theta.$$

Putting it all together, we obtain the torque balance equation

$$I\dot{\omega} = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta. \tag{3}$$

This is called an *integro-differential equation* because it involves both derivatives and integrals.

Amplitude Equations

Equations (2) and (3) completely specify the evolution of the system. Given the current values of $m(\theta,t)$ and $\omega(t)$, (2) tells us how to update m and (3) tells us how to update ω. So no further equations are needed.

If (2) and (3) truly describe the waterwheel's behavior, there must be some pretty complicated motions hidden in there. How can we extract them? The equations appear much more intimidating than anything we've studied so far.

A miracle occurs if we use Fourier analysis to rewrite the system. Watch!

Since $m(\theta,t)$ is periodic in θ, we can write it as a Fourier series

$$m(\theta,t) = \sum_{n=0}^{\infty} \left[a_n(t)\sin n\theta + b_n(t)\cos n\theta \right]. \tag{4}$$

By substituting this expression into (2) and (3), we'll obtain a set of ***amplitude equations***, ordinary differential equations for the amplitudes a_n, b_n of the different *harmonics* or *modes*. But first we must also write the inflow as a Fourier series:

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta. \tag{5}$$

There are no $\sin n\theta$ terms in the series because water is added *symmetrically* at the top of the wheel; the same inflow occurs at θ and $-\theta$. (In this respect, the waterwheel is unlike an ordinary, real-world waterwheel where asymmetry is used to drive the wheel in the same direction at all times.)

Substituting the series for m and Q into (2), we get

$$\begin{aligned}\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right] = &-\omega\frac{\partial}{\partial\theta}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right] \\ &+\sum_{n=0}^{\infty} q_n \cos n\theta \\ &-K\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right].\end{aligned}$$

Now carry out the differentiations on both sides, and collect terms. By orthogonality of the functions $\sin n\theta$, $\cos n\theta$, we can equate the coefficients of each harmonic separately. For instance, the coefficient of $\sin n\theta$ on the left-hand side is $\dot{a}_n$, and on the right it is $n\omega b_n - Ka_n$. Hence

$$\dot{a}_n = n\omega b_n - Ka_n. \tag{6}$$

Similarly, matching coefficients of $\cos n\theta$ yields

$$\dot{b}_n = -n\omega a_n - Kb_n + q_n. \tag{7}$$

Both (6) and (7) hold for all $n = 0, 1, \ldots$.

Next we rewrite (3) in terms of Fourier series. *Get ready for the miracle.* When we substitute (4) into (3), only one term survives in the integral, by orthogonality:

$$\begin{aligned} I\dot{\omega} &= -\nu\omega + gr\int_0^{2\pi}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right]\sin\theta\, d\theta \\ &= -\nu\omega + gr\int_0^{2\pi} a_1 \sin^2\theta\, d\theta \\ &= -\nu\omega + \pi g r a_1. \end{aligned} \tag{8}$$

Hence, only a_1 enters the differential equation for $\dot{\omega}$. But then (6) and (7) imply that a_1, b_1, *and* ω *form a closed system—these three variables are decoupled from all the other* a_n, b_n, $n \neq 1$! The resulting equations are

$$\begin{aligned} \dot{a}_1 &= \omega b_1 - Ka_1 \\ \dot{b}_1 &= -\omega a_1 - Kb_1 + q_1 \\ \dot{\omega} &= (-\nu\omega + \pi g r a_1)/I. \end{aligned} \tag{9}$$

(If you're curious about the higher modes a_n, b_n, $n \neq 1$, see Exercise 9.1.2.)

We've simplified our problem tremendously: the original pair of integro-partial differential equations (2), (3) has boiled down to the three-dimensional system (9). It turns out that (9) is equivalent to the Lorenz equations! (See Exercise 9.1.3.) Before we turn to that more famous system, let's try to understand a little about (9). No one has ever *fully* understood it—its behavior is fantastically complex—but we can say something.

Fixed Points

We begin by finding the fixed points of (9). For notational convenience, the usual asterisks will be omitted in the intermediate steps.

Setting all the derivatives equal to zero yields

$$a_1 = \omega b_1 / K \tag{10}$$
$$\omega a_1 = q_1 - K b_1 \tag{11}$$
$$a_1 = \nu\omega / \pi g r . \tag{12}$$

Now solve for b_1 by eliminating a_1 from (10) and (11):

$$b_1 = \frac{K q_1}{\omega^2 + K^2} . \tag{13}$$

Equating (10) and (12) yields $\omega b_1 / K = \nu\omega / \pi g r$. Hence $\omega = 0$ or

$$b_1 = K\nu / \pi g r . \tag{14}$$

Thus, there are two kinds of fixed point to consider:

1. If $\omega = 0$, then $a_1 = 0$ and $b_1 = q_1 / K$. This fixed point

$$(a_1{*}, b_1{*}, \omega{*}) = (0,\ q_1/K,\ 0) \tag{15}$$

corresponds to a state of *no rotation*; the wheel is at rest, with inflow balanced by leakage. We're not saying that this state is stable, just that it exists; stability calculations will come later.

2. If $\omega \neq 0$, then (13) and (14) imply $b_1 = K q_1 / (\omega^2 + K^2) = K\nu / \pi g r$. Since $K \neq 0$, we get $q_1 / (\omega^2 + K^2) = \nu / \pi g r$. Hence

$$(\omega{*})^2 = \frac{\pi g r q_1}{\nu} - K^2 . \tag{16}$$

If the right-hand side of (16) is positive, there are two solutions, $\pm\omega{*}$, corresponding to *steady rotation* in either direction. These solutions exist if and only if

$$\frac{\pi g r q_1}{K^2 \nu} > 1 . \tag{17}$$

The dimensionless group in (17) is called the ***Rayleigh number***. It measures how hard we're driving the system, relative to the dissipation. More precisely, the ratio in (17) expresses a competition between g and q_1 (gravity and inflow, which tend to spin the wheel), and K and ν (leakage and damping, which tend to stop the wheel). So it makes sense that steady rotation is possible only if the Rayleigh number is large enough.

The Rayleigh number appears in other parts of fluid mechanics, notably convection, in which a layer of fluid is heated from below. There it is proportional to the difference in temperature from bottom to top. For small temperature gradients, heat is conducted vertically but the fluid remains motionless. When the Rayleigh number increases past a critical value, an instability occurs—the hot fluid is less dense and begins to rise, while the cold fluid on top begins to sink. This sets up a pattern of convection rolls, completely analogous to the steady rotation of our waterwheel. With further increases of the Rayleigh number, the rolls become wavy and eventually chaotic.

The analogy to the waterwheel breaks down at still higher Rayleigh numbers, when turbulence develops and the convective motion becomes complex in space as well as time (Drazin and Reid 1981, Bergé et al. 1984, Manneville 1990). In contrast, the waterwheel settles into a pendulum-like pattern of reversals, turning once to the left, then back to the right, and so on indefinitely (see Example 9.5.2).

9.2 Simple Properties of the Lorenz Equations

In this section we'll follow in Lorenz's footsteps. He took the analysis as far as possible using standard techniques, but at a certain stage he found himself confronted with what seemed like a paradox. One by one he had eliminated all the known possibilities for the long-term behavior of his system: he showed that in a certain range of parameters, there could be no stable fixed points and no stable limit cycles, yet he also proved that all trajectories remain confined to a bounded region and are eventually attracted to a set of zero volume. What could that set be? And how do the trajectories move on it? As we'll see in the next section, that set is the strange attractor, and the motion on it is chaotic.

But first we want to see how Lorenz ruled out the more traditional possibilities. As Sherlock Holmes said in *The Sign of Four*, "When you have eliminated the impossible, whatever remains, however improbable, must be the truth."

The Lorenz equations are

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz. \end{aligned} \qquad (1)$$

Here σ, r, $b > 0$ are parameters. σ is the ***Prandtl number***, r is the Rayleigh number, and b has no name. (In the convection problem it is related to the aspect ratio of the rolls.)

Nonlinearity

The system (1) has only two nonlinearities, the quadratic terms xy and xz. This should remind you of the waterwheel equations (9.1.9), which had two nonlinearities, ωa_1 and ωb_1. See Exercise 9.1.3 for the change of variables that transforms the waterwheel equations into the Lorenz equations.