Homework 1
Stability analysis of non-linear dynamical systems

(Max score: 100)

15-382: Collective Intelligence (Spring 2018)

OUT: January 21, 2018, at 3:00pm

DUE: January 28, 2018 at 9:am - Available late days: 1

Instructions

Homework Policy

Homework is due on Autolab by the posted deadline. As a general rule, you have a total of 6 late days. For this homework you cannot use more than 1 late day. No credit will be given for homework submitted after the late day. After your 6 late days have been used you will receive 20% off for each additional day late.

If you find a solution in any source other than the material provided, you must mention the source.

Submission

Create a zipped archive including: a PDF file with the answers to the provided questions (they can be handwritten, but in this case you must have / use a "readable" handwriting), files that have been used for dealing with the questions that require programming, a README file that specifies how to use / run the programming files. The zipped archive should be submitted to Homework 1 on autolab.

Contents

1 Analysis of a dynamical model of competing species

In some closed environment there are *two different populations*, 1 and 2, that do not mate with each other (e.g., different animal species, closed human groups/societies, different capital assets, antigens-antibodies immune systems).

In absence of mutual interaction, we have seen that a reasonable model for describing the dynamical evolution of the system of two populations (in terms of how many individuals are there), is to individually apply a *logistic growth* model, that accounts for the capacity of the environment in terms of how many individuals of each type it can contain. In this case, the dynamical system that models populations' evolution over time is:

$$\begin{cases} \frac{dx_1}{dt} = x_1(g_1 - c_1 x_1) \\ \frac{dx_2}{dt} = x_2(g_2 - c_2 x_2) \end{cases} \tag{1}$$

where, g_1, g_2 are the (free capacity) growth rates, and c_1, c_2, are the capacity limits imposed by the environment (e.g., because of a limited food supply, or capital, or energy).

If the two populations *compete* for a common, vital resource, the logistic model is not sufficient to describe systems' evolution. However, it can still be used by adding a non-linear *correction* that accounts for the negative interference between the populations, obtaining the following new model:

$$\begin{cases} \frac{dx_1}{dt} = x_1(g_1 - c_1x_1 - i_1x_2) \\ \frac{dx_2}{dt} = x_2(g_2 - c_2x_2 - i_2x_1) \\ \end{cases} \tag{2}$$

where the coefficient i_1 represents the negative impact that the presence of population 2 has on population 1, and, symmetrically, i_2 is the negative impact that the presence of population 2 has on population 1. More precisely, they define the rate of decrease of one population proportionally to the current magnitude of the other population. All coefficients are positive (that is necessary to make fully sense of the negative signs).

As defined, the model is clearly non-linearly, that can pose challenges finding analytic solutions and studying the stability of the equilibrium points. However, in order to control the overall stability of the environment, it is of *strategic importance* to get a good understanding of what dynamics are to be expected, and how it's potentially possible to intervene on the system (e.g., to avoid that one animal species disappears from an ecosystem).

At this aim, a number of tools for the analytic and numeric analysis of linear and non linear systems of ODEs are provided and *must be used* for the homework. The tools are in the form of a Python class, `DynamicalSystem`, that is included (together with examples of use) in the provided file `ode-analysis.py`. Class methods allow to: solve an ODE system (both symbolically and numerically), compute eigenvalues and eigenvectors of linear systems, find critical points, compute Jacobians (both symbolically and numerically), plot vector and flow fields.

1.1 Analysis of a specific instance of the model (50 points)

Given the following instance of the general model above.

$$\begin{cases} \frac{dx_1}{dt} = x_1(1 - x_1 - \frac{1}{3}x_2) \\ \frac{dx_2}{dt} = x_2(\frac{3}{4} - x_2 - \frac{1}{2}x_1) \end{cases}$$

Let's assume that the model describes two populations of two distinct animal species competing for food supply in a (mostly) closed environment (e.g., a pond).

Your task is to make a complete study of the system, that includes:

1. Find analytic solutions (if you can't make it, this is not necessary for the overall analysis);
2. Find the critical points;
3. Study the characteristics of stability of the critical points using a linearization approach (that, in turn, requires to linearize the system, and study the eigenspaces of the resulting linear system).
4. Plot both the vector fields and the flows of the system (plots must be well readable and include all the relevant information).
5. Define the equations of the nullclines. Specify what is their geometric relationship with trajectory flows.
6. For each critical point: (i) discuss the geometry of the phase space around it and the impact that the geometry has on the local evolution of the trajectories; (ii) discuss the "ecological meaning" of the previous discussion for what concerns the dynamics of the two animal species (e.g., "the equilibrium point $(0, 1)$ means that population 1 gets extinct, and according to the geometry of the trajectories, this solution happens for all trajectories starting within region A of the phase space, the velocity of convergence is however slow because of the elliptic form of the curves, . . . ")

1.2 Analysis of the parametric form of the model (50 points)

Let's consider the system in its general form, as provided in Eq. 2. The goal is to define its stability properties depending on the choice of the parameters.

1. Find the critical points;
2. Study the characteristics of stability of the critical points (using a linearization approach) vs. the parameters;
3. Define the parametric equations of the nullclines and discuss their geometry. Specify what is their geometric relationship with trajectory flows.
4. Discuss the "ecological meaning" of certain parameter choices (e.g., :"when the mutual interference between the two populations is lower that the growth rates, than a stable solution can arise, otherwise one of the two populations get extinct").