Homework 5

Classical and Evolutionary Game Theory

(Max useful score: 100 - Available points: 150)

15-382: Collective Intelligence (Spring 2018)

OUT: April 15, 2018, at 11:00am

DUE: April 19, 2018 at 11:00pm - Available late days: 1

Instructions

Homework Policy

You must try to answer to at least two questions from Section 1 and two questions from Section 2.

Homework is due on Autolab by the posted deadline. As a general rule, you have a total of 6 late days. For this homework you cannot use more than 1 late day. No credit will be given for homework submitted after the late days. After your 6 late days have been used you will receive 20% off for each additional day late.

If you find solutions in any source other than the material provided, you must mention the source.

Submission

Create a zipped archive including: a PDF file with the answers to the provided questions (they can be handwritten, but in this case you must have / use a "readable" handwriting). The zipped archive should be submitted to Homework 5 on Autolab.

Contents

1 Classical Game theory (64 points)

1.1 Tosca's game (10 points)

In Puccini's opera *Tosca*, Tosca's lover has been condemned to death. The police chief, Scarpia, offers to *fake* the execution if Tosca will *sleep* with him. The bargain is struck. However, since Tosca has a preference to keep her honor, at the meeting she stabs and *kills* Scarpia. Unfortunately, Scarpia had a preference towards his *duty* as a policeman, and had secretly ordered to execute Tosca's lover.

1. Construct a game theoretic representation of the opera plot, representing it in normal form.
2. Assign payoffs such that the only equilibrium of the game corresponds to the opera's outcome.

1.2 Prisoner's dilemmas and dominant strategies (11 points)

A Prisoner's Dilemma game model occurs when: (i) each player has a strategy that strictly dominates all his other strategies, but (ii) each player has another strategy such that, if all players were to use this alternative, all players would receive higher payoffs than those they receive when they all use their dominant strategies. Unfortunately, because of dominance, the two players do not choose the second strategy without some additional mechanism that would prime coordinated cooperation.

One type of games that present the characteristics of prisoner's dilemmas are the so-called *public goods* games. In a public goods game, when a player cooperates, he adds more to the total payoffs of all players than his cost of cooperating, but his cost of cooperating is greater than his individual share of the payoffs. Public goods games are one type of *social dilemma*. In a social dilemma, all players gain when all cooperate, but each has an incentive to defect, which will give him a gain at the expense of the others.

Let's consider a simple model of *global warming scenario* as a public goods game. There are ten countries considering fighting global warming. Each country must choose to spend an amount x_i to reduce its carbon emissions, where $0 \leq x_i \leq 1$. The total benefits produced by these expenditures equal twice the total expenditures: $2(x_1 + x_2 + \ldots + x_{10})$. Each country receives $\frac{1}{10}$ of the total benefits.

1. For each country, find the strategy that dominates all the other strategies. Show the computations / reasoning.
2. What would be the payoff of a country i if it decides to go for $x_i = 1$, the strategy for full spending for fighting against global warming?
3. Unfortunately, even if $x_i = 1$ is a good deal for country i, the country would always be tempted to cheat, reducing its expenditure on global warming. Show and quantify why this is the case.

1.3 Inheritance tricks (11 points)

A man, an ex-professor of computer science, has two sons, that always have had some conflicts between each other. Instead, the professor would like to see his sons to cooperate with each other. Therefore, in his will it states that the two sons must each specify a sum of money, s_i, that they are willing to accept for inheritance when he dies. The value of his estate is 1,000,000 USD, which is then the available inheritance.

If $s_1 + s_2 \leq 1,000,000$, then each son gets the sum he asked for and the reminder (if any) goes to the charity fund for retired computer science professors. If $s_1 + s_2 > 1,000,000$, then neither son receives any money and the entire 1,000,000 goes to retired professors.

Find *all* the pure strategy Nash equilibria of the game assuming that the two men only care about their share of inheritance, they do not talk to each other, and can only ask for integer amounts.

1.4 Convex combination of correlated equilibria (16 points)

Let $p_1, \ldots, p_n$ be probability distributions representing n different correlated equilibria in a 2-player game. Prove that any convex combination of $p_1, \ldots, p_n$ is also a correlated equilibrium.

1.5 Equality of payoffs at mixed Nash equilibrium (16 points)

Let (σ^*, σ^*) be a mixed Nash equilibrium in a two-player game, let S be the finite set of pure strategies, and let S_1^* be the support of σ_1^* (σ_1^* is the mixed strategy adopted by player 1 at the Nash equilibrium).

1. Formally prove that $\pi_1(s, \sigma_2^*) = \pi_1(\sigma_1^*, \sigma_2^*),\ \forall s \in S_1^*$, where π_1 is the expected payoff of player 1.
2. Discuss, possibly using a simple example, why the above statement is "intuitively" true.

2 Evolutionary game theory (81 points)

2.1 Nash implies a fixed point in the replicator dynamics (16 points)

Formally prove that if (σ^*, σ^*) is a symmetric Nash equilibrium, then the population state $\boldsymbol{x}^* = \sigma^*$ is a fixed point of the replicator dynamics.

1. Prove the statement in the case the Nash equilibrium strategy σ^* is a *pure* strategy.
2. Prove the statement in the general case when the Nash equilibrium strategy σ^* is a *mixed* strategy. At this aim, make use of the statement of question 1.5.

2.2 Bird nesting choices (20 points)

A population of birds is strategically distributed in the environment so that in any different patch that has two trees suitable for nesting (T_1 and T_2) there are only two females. If the two females adopt the behavior of picking the same nesting site in the patch, then they each raise 2 offspring. If their behavior is to choose different sites, then they only raise 1 offspring each because they are more vulnerable to predators. The situation can be modeled as pairwise contest game which is repeatedly played in the bird population.

1. Model the situation as an evolutionary pairwise contest game and construct the 2-player payoff table.
2. Find all the symmetric pure Nash equilibria of the pairwise contest game.
3. Find all the symmetric mixed Nash equilibria of the pairwise contest game [*hint:* make use of the results of question 1.5 to rapidly find the answer].
4. Determining which of the Nash equilibria correspond to ESSs in the associated population game.
5. Derive the one-dimensional replicator dynamics equation and show that only the fixed points that correspond to an ESS are evolutionary end points.

Report the computations / reasoning for finding the results to the above questions!

2.3 Social networks (20 points)

Let's assume that there are two main platforms for social networks: W and L. A user of platform W has a basic utility of 1 (whatever this means for the user), while a user of platform L has a basic utility of 2, since L has better interface and more tools. If two users adopt the same platform, they are in the same social network, and therefore they can communicate over the network. A user's utility rises linearly with the proportion of people using the same platform, since he/she can communicate with more people. However, the utility grows up to a maximum increment of 2 (saturation effect). Let x be the proportion of W users, then the following payoffs can be defined for a game against the field (the users in the available social networks) $\pi(W, x) = 1 + 2x,\ \pi(L, x) = 2 + 2(1 - x)$.

The ESSs candidates in the population game are: two pure population strategies, σ_W and σ_L, and one mixed strategy, σ_m. In σ_W everyone uses W, in σ_L everyone uses L, and in σ_m W is used $\frac{3}{4}$ of the time.

Using the notion of post-entry population profile $\boldsymbol{x}_\varepsilon$ and the definition of ESS, find whether σ_W, σ_L, and σ_m are ESSs or not.

2.4 Asymptotically stable strategies in the Rock-Paper-Scissor population (25 points)

Let's consider the pairwise contest game with the payoffs given by the Paper-Rock-Scissors game where x_1 represents the fraction of individuals playing pure strategy R, x_2 is the fraction of individuals playing pure strategy P, and x_3 represents the fraction of individuals playing pure strategy S. We have seen that the polymorphic population $\boldsymbol{x}^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is a fixed point in the replicator dynamics but is not asymptotically stable. Define a variation of the RPS payoffs for the draw cases such that $\boldsymbol{x}^*$ is asymptotically stable in the new replicator dynamics.

1. Report the new payoff matrix and discuss the rational behind the changes.
2. Write down the replicator dynamics system.
3. Write down the replicator dynamics system in the reduced state (two state variables).
4. Study the replicator dynamics: identify all the fixed points and their stability properties.
5. Show the phase portrait of the system.
6. Describe in words what happens over time to a polymorphic population that starts in state $\boldsymbol{x} = (0.6, 0.2, 0.2)$.
7. Give an example scenario to where the RPS population game could apply.