15-382 COLLECTIVE INTELLIGENCE – S18

LECTURE 8:
DYNAMICAL SYSTEMS 7

INSTRUCTOR:
GIANNI A. DI CARO

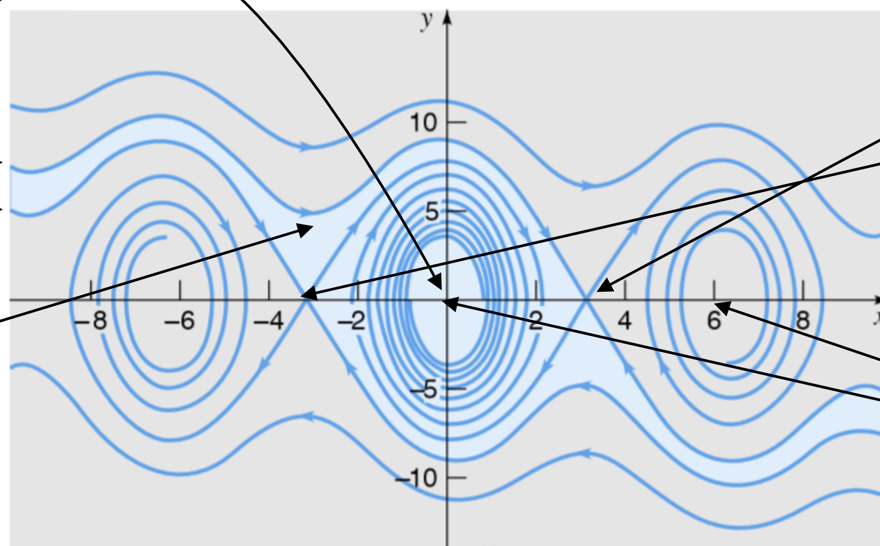
Geometries in the phase space

- Damped pendulum $dx/dt = y, \qquad dy/dt = -9\sin x - \frac{1}{5}y$

One cp in the region between two separatrix

Separatrix

Basin of attraction

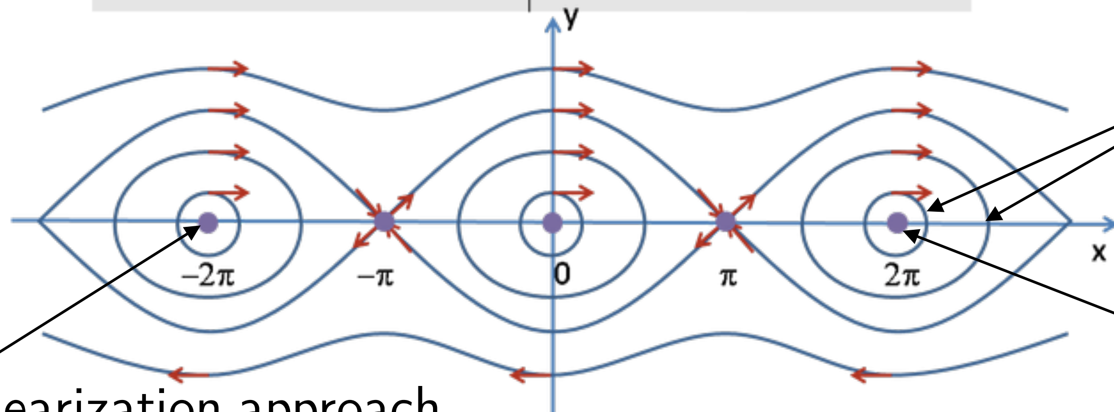


Saddle

Asymptotically unstable

Asymptotically stable spiral (or node)

- Undamped pendulum



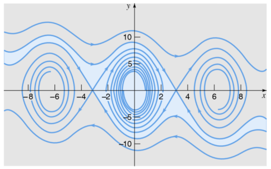
Closed orbits (periodic)

Fixed point (any period)

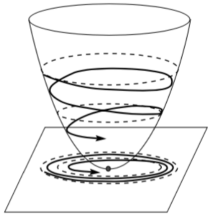
Center: the linearization approach doesn't allow to say much about stability

Geometries in the phase space ...

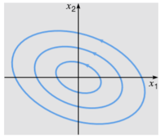
- **Question 1:** The linearization approach for studying the stability of critical points is a purely local approach. Going more global, what about the basin of attraction of a critical point?

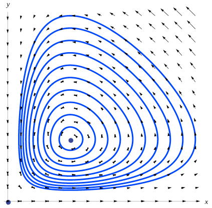


Lyapounov functions

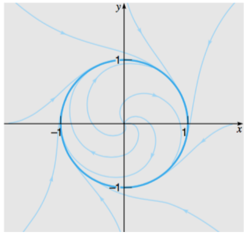


- **Question 2:** When the *linearization approach fails* as a method to study the stability of a critical point, can we rely on something else?





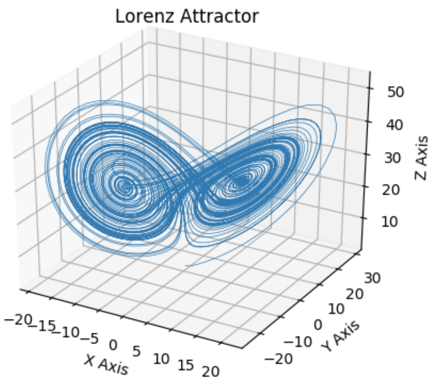
- **Question 3:** Are critical points and *well separated closed orbits* all the *geometries* we can have in the phase space?



Limit cycles

- **Question 4:** Does the dimensionality of the phase space impact on the possible geometries and limiting behavior of the orbits?

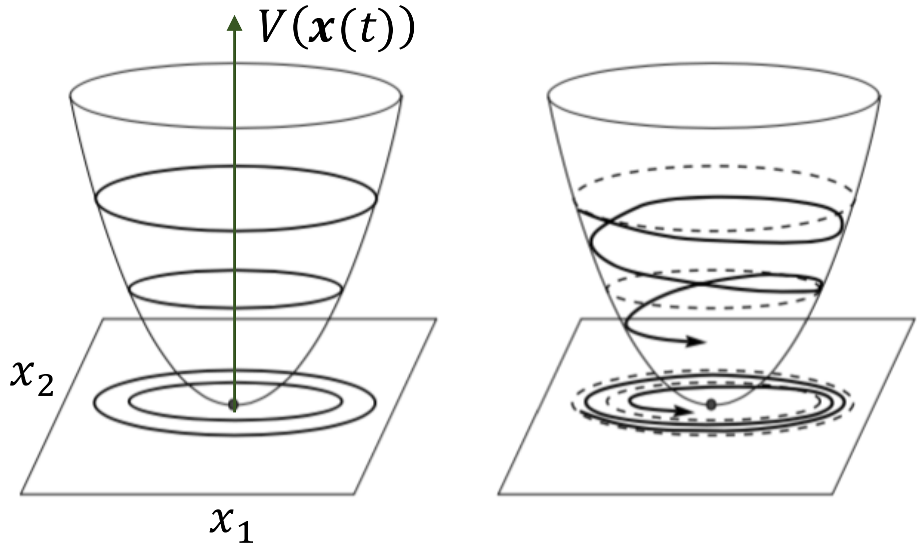
- **Question 5:** Are critical points and closed orbits the only forms of *attractors* in the dynamics of the phase space? Is *chaos* related to this?



Lyapunov direct method: Potential functions

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^n$$

$V(\boldsymbol{x}(t))$ = *Potential energy* of the system when in state $\boldsymbol{x}$, $V: \mathbb{R}^n \to \mathbb{R}$



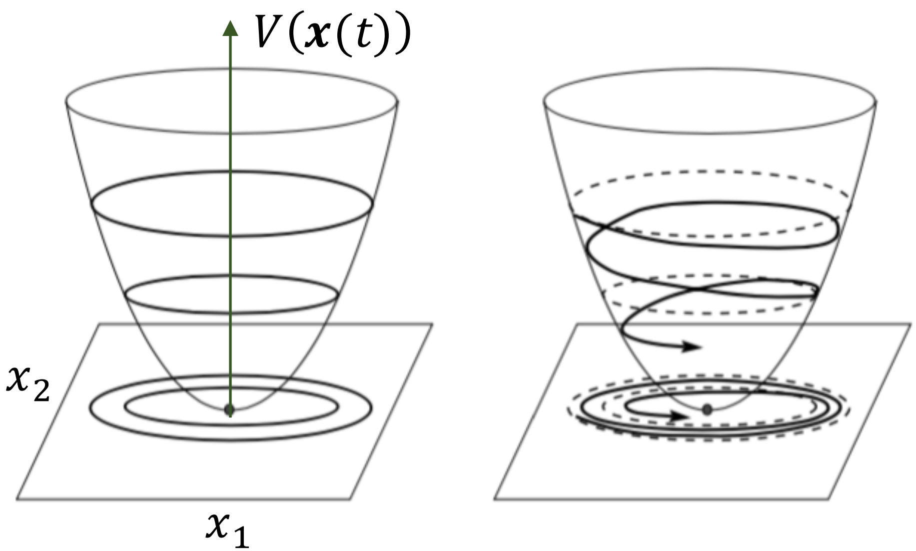
- Time rate of change of $V(\boldsymbol{x}(t))$ along a solution trajectory $\boldsymbol{x}(t)$, we need to take the derivative of V with respect to t. Using the chain rule:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial V}{\partial x_n}\frac{dx_n}{dt} = \frac{\partial V}{\partial x_1} f_1(x_1, \ldots, x_n) + \cdots + \frac{\partial V}{\partial x_n} f_n(x_1, \ldots, x_n)$$

Solutions do not appear, only the system itself!

LYAPUNOV FUNCTIONS

- $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^n$
- $\boldsymbol{x}^e$ equilibrium point of the system
- A function $V: \mathbb{R}^n \to \mathbb{R}$ continuously differentiable is called a **Lyapunov function** for $\boldsymbol{x}^e$ if for some neighborhood D of $\boldsymbol{x}^e$ the following hold:
  1. $V(\boldsymbol{x}^e) = 0$, and $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}^e$ in D
  2. $\dot{V}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x}$ in D
- If $\dot{V}(\boldsymbol{x}) < 0$, it's called a **strict Lyapunov function**



- $V(\boldsymbol{x}(t)) =$ *Energy* of the system when in state $\boldsymbol{x}$

1. $\boldsymbol{x}^e$ is a the bottom of the graph of the Lyapunov function
2. Solutions can't move up, but can only move down the side of the potential hole or stay level

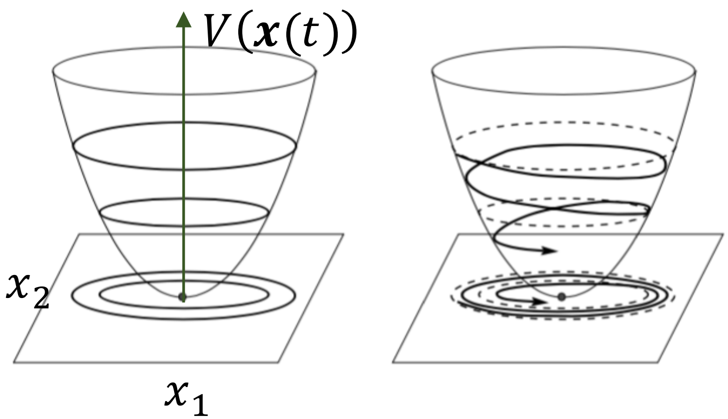
Lyapunov stability theorem

- **Theorem** (Sufficient conditions for stability):

  Let $\boldsymbol{x}^e$ be an (*isolated*) equilibrium point of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

  If there exists a *Lyapunov function* for $\boldsymbol{x}^e$, then $\boldsymbol{x}^e$ is stable.

  If there exists a *strict Lyapunov function* for $\boldsymbol{x}^e$, then $\boldsymbol{x}^e$ is asymptotically stable



- Any set D on which V is a strict Lyapunov function for $\boldsymbol{x}^e$ is a subset of the basin $B(\boldsymbol{x}^e)$
- If there exists a strict Lyapunov function, then there are no closed orbits in the region of its basin of attraction

- *Definition:* Let $\boldsymbol{x}^e$ be an asymptotically stable equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Then the **basin of attraction** of $\boldsymbol{x}^e$, denoted $B(\boldsymbol{x}^e)$, is the set of initial conditions $\boldsymbol{x}_0$ such that $\lim_{t\to\infty} \boldsymbol{F}(\boldsymbol{x}_0, t) = \boldsymbol{x}^e$

How do we define Lyapunov functions?

- **Physical systems:** Use the energy function of the system itself

For a damped pendulum $(x = \theta, y = \frac{d\theta}{dt})$ $V(x, y) = mgL(1 - \cos x) + \frac{1}{2}mL^2y^2.$

- **Other systems:** Guess!

$$dx/dt = -x - xy^2$$
$$dy/dt = -y - x^2y$$

$$\boldsymbol{x}^e = (0,0)$$

$$V(x, y) = ax^2 + bxy + cy^2$$

The function

$$V(x, y) = ax^2 + bxy + cy^2$$

is positive definite if, and only if,

$$a > 0 \quad \text{and} \quad 4ac - b^2 > 0,$$

and is negative definite if, and only if,

$$a < 0 \quad \text{and} \quad 4ac - b^2 > 0.$$

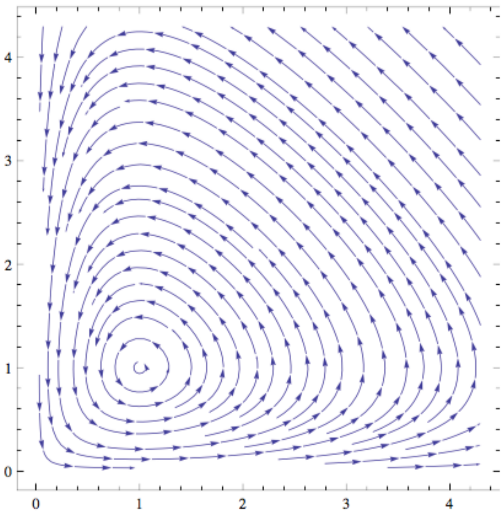
$$V_y(x, y) = bx + 2cy,$$
$$V_x(x, y) = 2ax + by,$$

$$\dot{V}(x, y) = (2ax + by)(-x - xy^2) + (bx + 2cy)(-y - x^2y)$$
$$= -[2a(x^2 + x^2y^2) + b(2xy + xy^3 + x^3y) + 2c(y^2 + x^2y^2)]$$

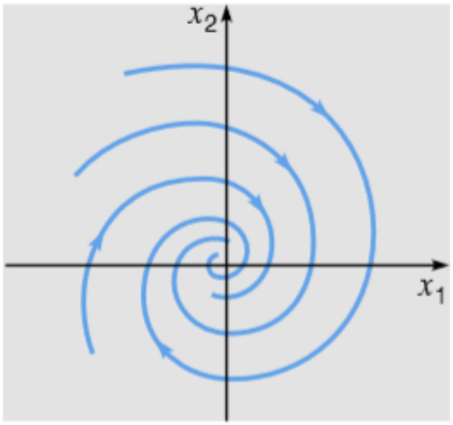
For $b = 0$, $a, c > 0$ $\rightarrow \dot{V} < 0, V > 0$ $\Rightarrow$ (0,0) is asymptotically stable

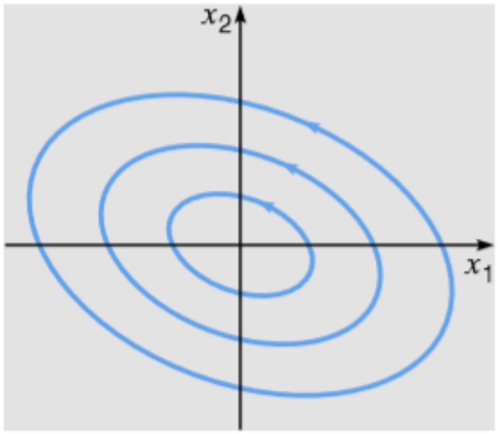
Limit cycles

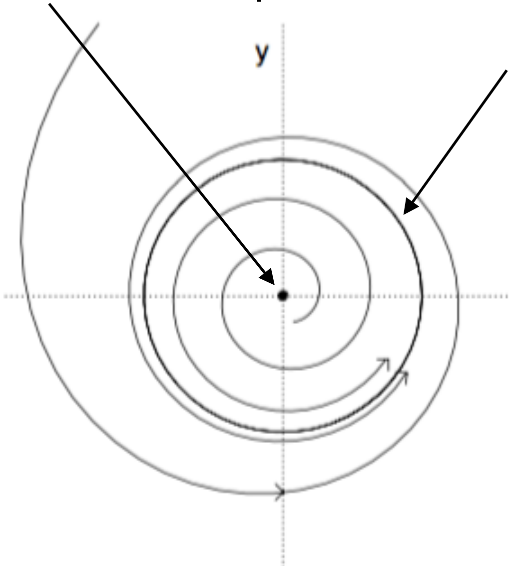
So far …



Unstable equilibrium

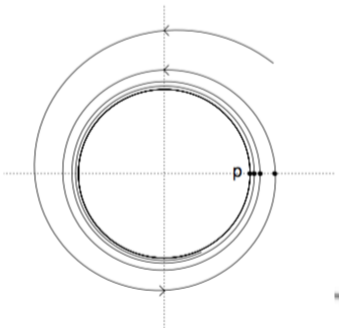


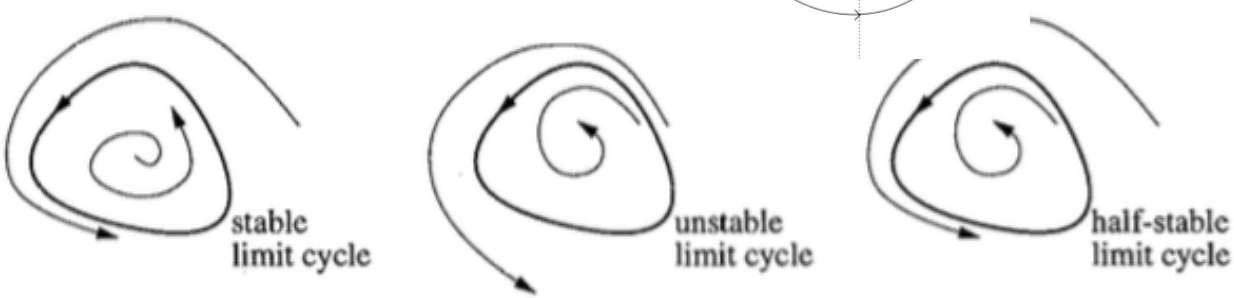




Periodic orbit: $\boldsymbol{x}(t+T) = \boldsymbol{x}(t)$

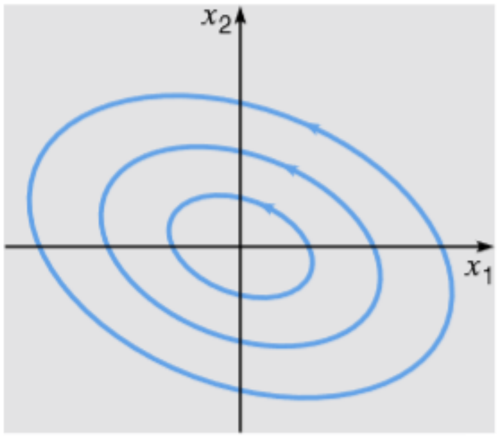
$\boldsymbol{\omega}$**-limit set** of points

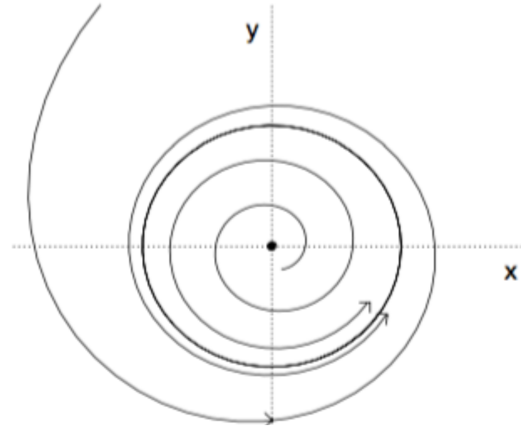


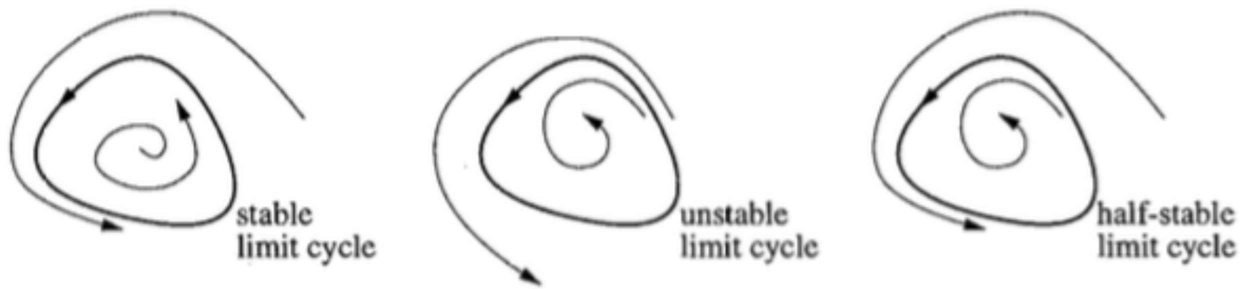


Something new: **limit cycles / orbital stability**

LIMIT CYCLES



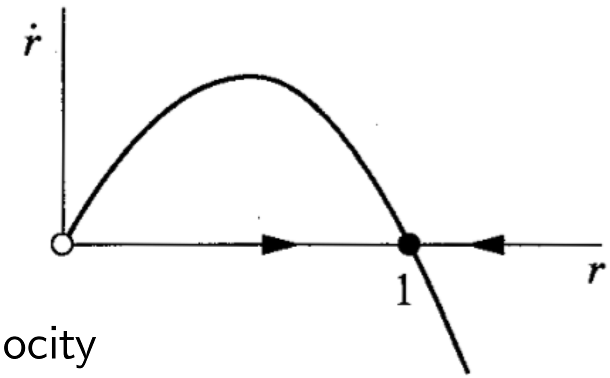




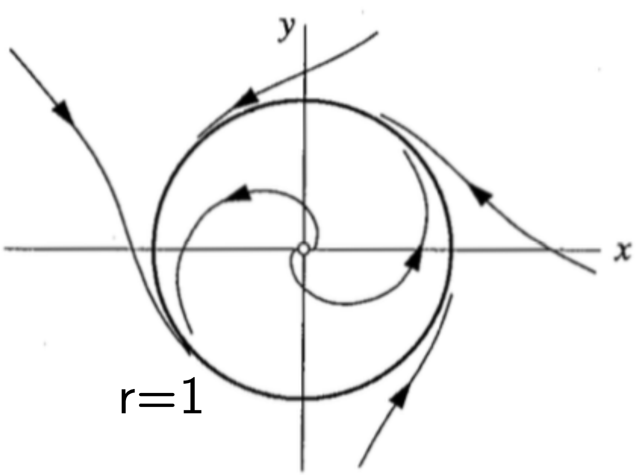
- A limit cycle is <u>an isolated closed trajectory</u>: neighboring trajectories are not close, they are spiral either away or to the cycle
- If all neighboring trajectories approach the limit cycle: **stable**, **unstable** otherwise, **half-stable** in mixed scenarios
- <u>In a linear system closed orbits are not isolated</u>

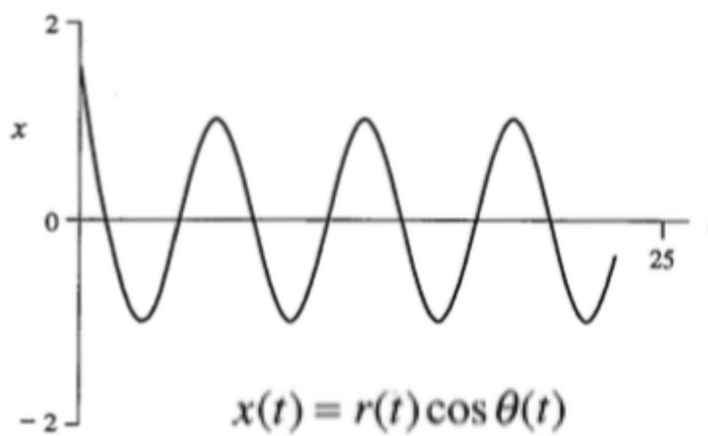
Limit cycle example

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases} \qquad r \geq 0$$



- Radial and angular dynamics are uncoupled, such that they can be analyzed separately
- The motion in θ is a rotation with constant angular velocity
- Treating $\dot{r} = r(1 - r^2)$ as vector field on the line, we observe that there are two critical points, (0) and (1)
- The phase space $(r, \dot{r})$ shows the functional relation: (0) is an unstable fixed point, (1) is stable, since the trajectories from either sides go back to $r = 1$





A solution component $x(t)$ starting outside unit circle ends to the circle (x oscillates with amplitude 1

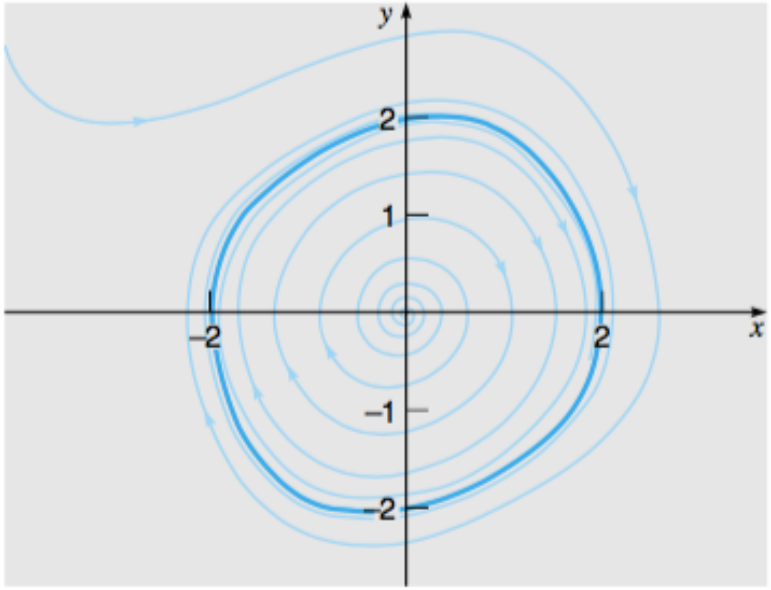
Van der Pol Oscillator

$$\underbrace{u'' + u}_{\text{Harmonic oscillator}} - \underbrace{\mu(1-u^2)u'}_{\text{Nonlinear damping}}$$

$$x' = v$$
$$y' = -x + \mu(1-x^2)y$$

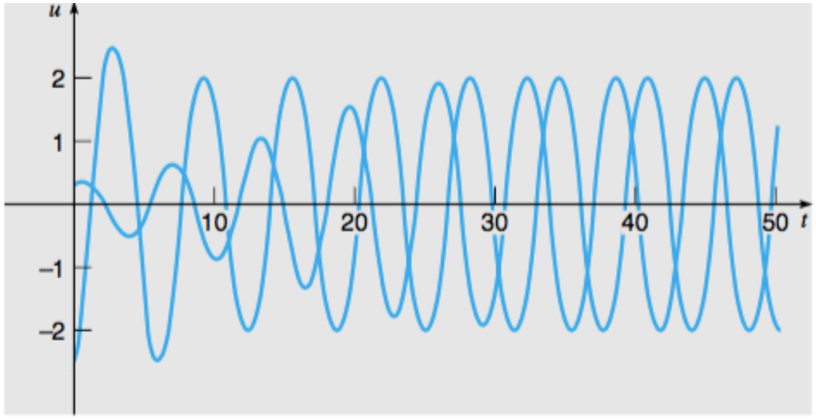
Nonlinear damping:

- Positive (regular) damping for $|u| > 1$ — Oscillations are large: it forces them to decay
- Negative (reinforcing) damping for $|u| < 1$ — Oscillations are small: it pumps them back



→ System settles into a self-sustained oscillation where the energy dissipated over one cycle balances the energy pumped in

→ **Unique limit cycle for each value of $\mu > 0$**



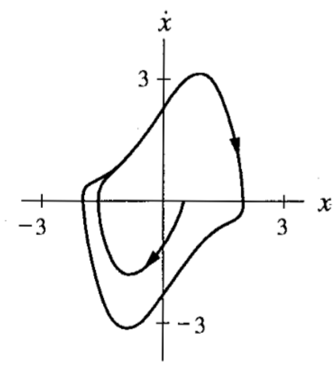
Two different initial conditions converge to the same limit cycle

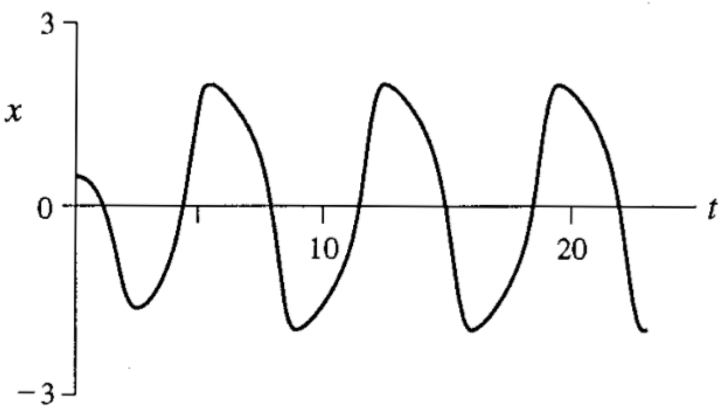
Van Der Pol Oscillator

$$u'' + u - \mu(1 - u^2)u'$$

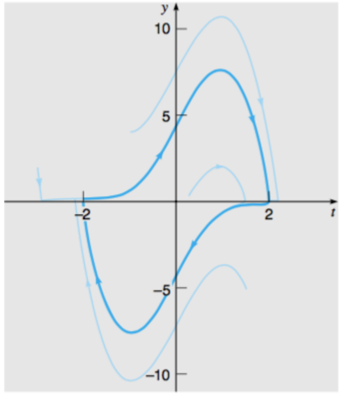
$$x' = v$$
$$y' = -x + \mu(1 - x^2)y$$

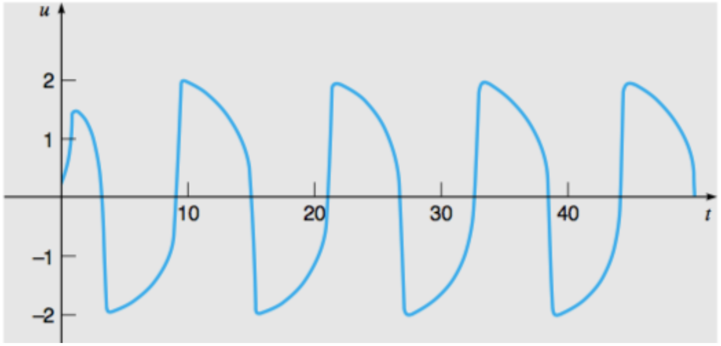
$\mu = 1.5$, starting from $(x, \dot{x}) = (0.5, 0)$ at $t = 0$





$\mu = 5$





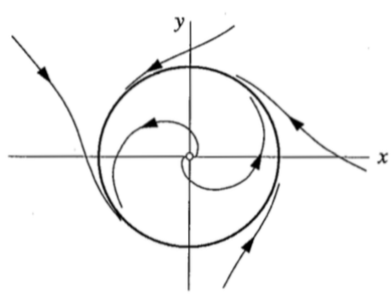
Numeric integration. Analytic solution is difficult

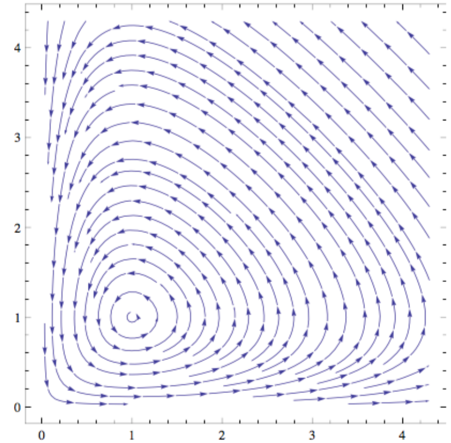
Conditions of existence of limit cycles

Under which conditions do close orbits / limit cycles exist?

We need a few preliminary results, in the form of the next two theorems, formulated for a two dimensional system: $\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$

- **Theorem** (Closed trajectories and critical points):
  Let the functions f_1 and f_2 have continuous first partial derivatives in a domain D of the phase plane.
  A closed trajectory of the system must necessarily enclose *at least one critical (equilibrium) point*.
  If it encloses only one critical point, the critical point cannot be a saddle point.



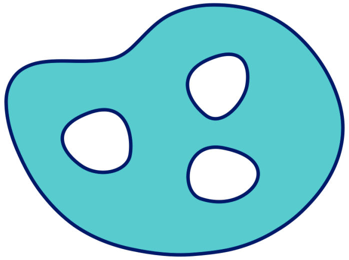


Exclusion version: if a given region contains no critical points, or only saddle points, then there can be no closed trajectory lying *entirely* in the region.

Conditions of existence of limit cycles

- **Theorem** (Existence of closed trajectories):
  Let the functions f_1 and f_2 have continuous first partial derivatives in a *simply connected domain* D of the phase plane.
  If $\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)_{x_1, x_2}$ has the same *sign* throughout D, then there is no closed trajectory of the system lying entirely in D

- If sign changes nothing can be said
- *Simply connected domains:*
  - A simply connected domain is a domain with no holes
  - In a simply connected domain, any path between two points can be continuously shrink to a point without leaving the set
  - Given two paths with the same end points, they can be continuously transformed one into the other while staying the in the domain



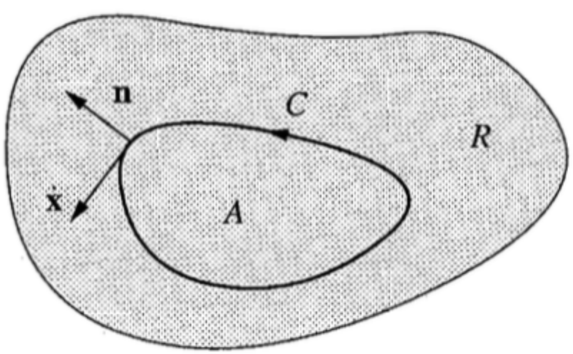
Not a simply connected domain

Proof of the theorem (only for fun)

- **Theorem** (Existence of closed trajectories):
  Let the functions f_1 and f_2 have continuous first partial derivatives in a *simply connected domain* R of the phase plane.
  If $\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)_{x_1,\, x_2}$ has the same *sign* throughout R, then there is no closed trajectory of the system lying entirely in R
- The proof is based on *Green's theorem*, a fundamental theorem in calculus: if C is a sufficiently smooth simple closed curve, and if F and G are two continuous functions and have continuous first partial derivatives, then:

$$\int_C [F(x,y)\,dy - G(x,y)\,dx] = \iint_R [F_x(x,y) + G_y(x,y)]\,dA,$$

where C is traversed counterclockwise and A is the region enclosed by C.



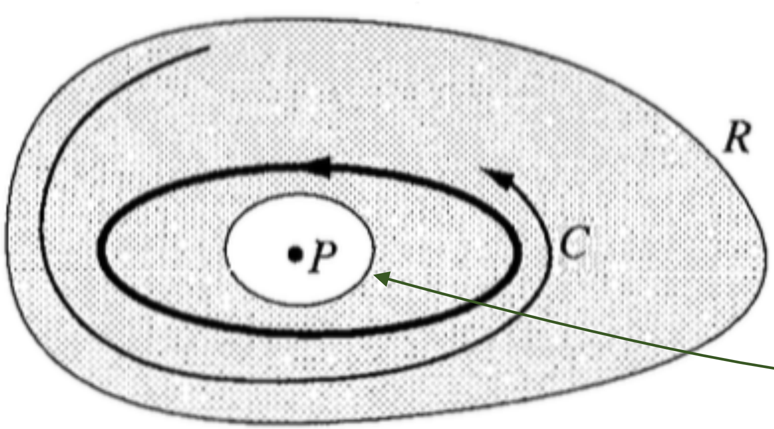
Let's suppose that C is a periodic solution and $F = f_1, G = f_2$, such that $F_x + F_y$ has the same sign in R. This implies that the double integral must be $\neq 0$. The line integral can be written as $\oint_C (\dot{x}_1, \dot{x}_2) \cdot \boldsymbol{n}\ d\ell$ which is zero, because C is a solution and the vector $(\dot{x}_1, \dot{x}_2)$ is always tangent to it → We get a contradiction.

POINCARE'-BENDIXSON THEOREM

- **Theorem** (Poincare'- Bendixson)

  Suppose that:
  - R is a closed, bounded subset of the phase plane
  - $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is a continuously differentiable vector field on an open set containing R
  - R does not contain any critical points
  - There exists a trajectory C that is confined in R, in the sense that is starts in R and stays in R for all future time

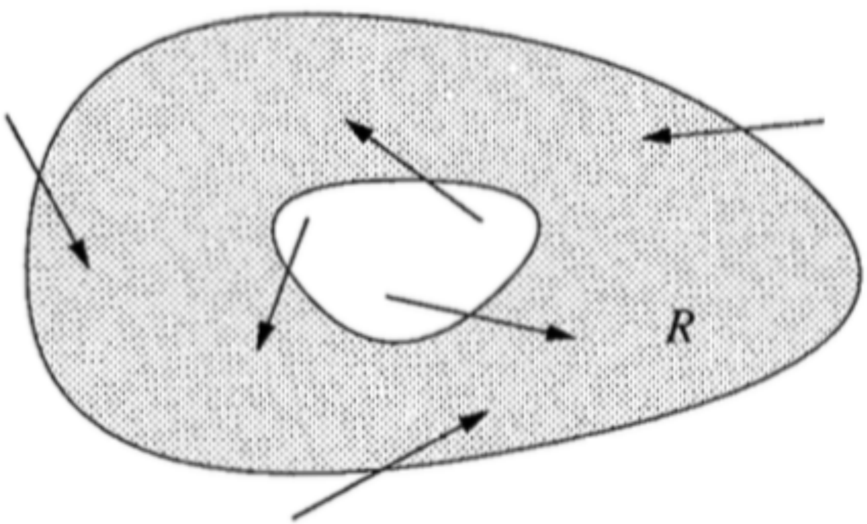
  Then, either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$, in either case R contains a closed orbit / periodic solution (and, possibly, a limit cycle)



Remark: If R contains a closed orbit, then, because of the previous theorem, it must contain a critical point $P \Longrightarrow$ R cannot be simply connected, it must have a *hole*

Poincare'-Bendixson Theorem

- How do we verify the conditions of the theorem in practice?
- ✓ R is a closed, bounded subset of the phase plane
- ✓ $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is a continuously differentiable vector field on an open set containing R
- ✓ R does not contain any critical points
- ❖ There exits a trajectory C that is confined in R, in the sense that is starts in R and stays in R for all future time: *Difficult one!*



1. Construct a **trapping region** R: a closed connected set such that the vector field points inward on the boundary of R → All trajectories are confined in R
2. If R can also be arranged to not include any critical point, the theorem guarantees the presence of a closed orbit

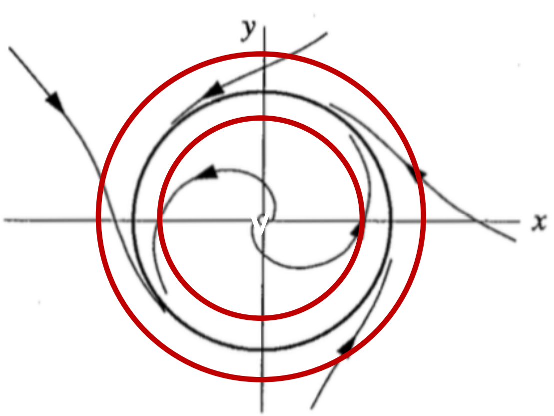
Checking P-B conditions

- It's difficult, in general

$$\begin{cases} \dot{r} = r(1-r^2) + \mu r\cos\theta \\ \dot{\theta} = 1 \end{cases} \qquad r \geq 0$$

For this system we saw that, for μ=0, $r = 1$ is a limit cycle. Is the cycle still present for $\mu > 0$, but small?

- In this case, we know where to look to verify the conditions of the theorem: let's find an annular region around the circle $r = 1$: $0 < r_{min} \leq r \leq r_{max}$, that plays the role of trapping region, finding r_{min} and r_{max} such that $\dot{r} < 0$ on the outer circle, and $\dot{r} > 0$ on the inner one
- The conditions of no fixed points in the annulus region is verified since $\dot{\theta} > 0$



- For $r = r_{min}$, $\dot{r}$ must be > 0: $r(1-r^2) + \mu r\cos\theta > 0$, observing that $\cos\theta \geq -1$, it's sufficient to consider $(1-r^2) + \mu > 0 \rightarrow r_{min} < \sqrt{1-\mu}$, $\mu < 1$
- A similar reasoning holds for r_{max}: $r_{max} > \sqrt{1+\mu}$
- The should be chosen as tight as possible
- Since all the conditions of the theorem as satisfied, a limit cycle exists for the selected r_{min}, r_{max}

Checking P-B conditions for Van Der Pol

- *A failing example*

$$u'' + u - \mu(1-u^2)u' \qquad \begin{aligned} x' &= v \\ y' &= -x + \mu(1-x^2)y \end{aligned} \qquad \text{Van Der Pol}$$

Critical point: origin, the linearized system has eigenvalues $(\mu \pm \sqrt{\mu^2 - 4})/2$

→ (0,0) is unstable spiral for $0 < \mu < 2$

→ (0,0) is an unstable node for $\mu \geq 2$

- Closed trajectories? The first theorem says that if they exist, they must enclose the origin, the only critical point. From the second theorem, observing that $\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) = \mu(1-x^2)$, if there are closed trajectories, they are not in the strip $|x| < 1$, where the sign of the sum is positive
- Neither the application of the P-B theorem is conclusive / easy → …

Checking P-B conditions for Van Der Pol

$$x' = y$$
$$y' = -x + \mu(1 - x^2)y$$

The application of the Poincaré–Bendixson theorem to this problem is not nearly as simple as for the preceding example. If we introduce polar coordinates, we find that the equation for the radial variable r is

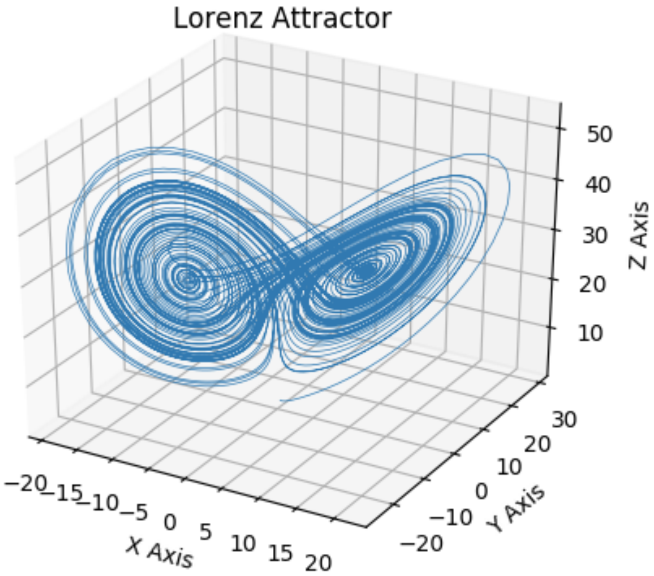
$$r' = \mu(1 - r^2 \cos^2\theta) r \sin^2\theta. \tag{21}$$

Again, consider an annular region R given by $r_1 \le r \le r_2$, where r_1 is small and r_2 is large. When $r = r_1$, the linear term on the right side of Eq. (21) dominates, and $r' > 0$ except on the x-axis, where $\sin\theta = 0$ and consequently $r' = 0$ also. Thus, trajectories are entering R at every point on the circle $r = r_1$ except possibly for those on the x-axis, where the trajectories are tangent to the circle. When $r = r_2$, the cubic term on the right side of Eq. (21) is the dominant one. Thus $r' < 0$ except for points on the x-axis where $r' = 0$ and for points near the y-axis where $r^2 \cos^2\theta < 1$ and the linear term makes $r' > 0$. Thus, no matter how large a circle is chosen, there will be points on it (namely, the points on or near the y-axis) where trajectories are leaving R. Therefore, the Poincaré–Bendixson theorem is not applicable unless we consider more complicated regions.

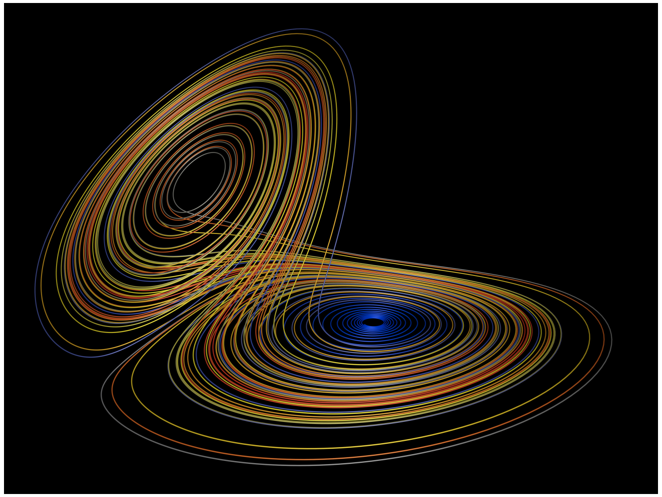
Poincare'-Bendixon Theorem: No *chaos* in 2D!

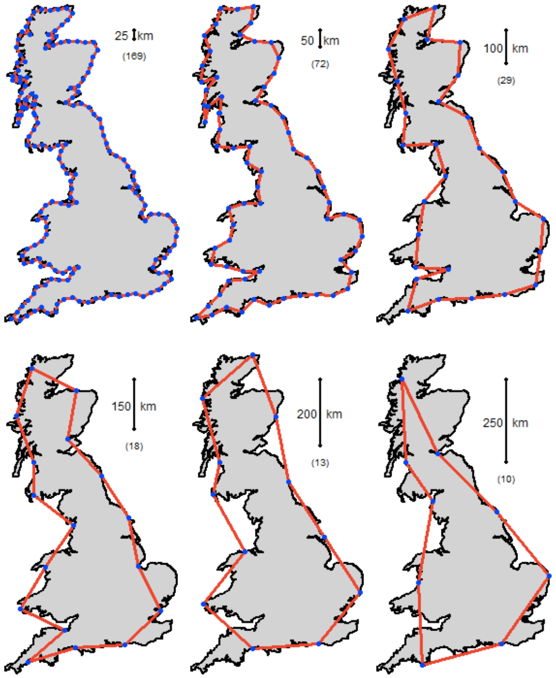
- Only apply to two-dimensional systems!
- It says that second-order (two-dimensional) dynamical systems are overall "well-behaved" and the dynamical possibilities are limited: if a trajectory is confined to a closed, bounded region that contains no equilibrium points, then the trajectory must eventually approach a closed orbit, nothing more complicated that this can happen
- A trajectory will either diverge, or settle down to a **fixed point** or a **periodic orbit / limit cycle**, that are the attractors of system's dynamics
- **What about higher dimensional systems, for $n \geq 3$?**
- *Trajectory may wonder around forever in a bounded region without settling down to a fixed point or a closed orbit!*
- In some cases the trajectories are attracted to a complex geometric objects called **strange attractor**, a fractal set on which the motion is **aperiodic** and **sensitive to tiny changes in the initial conditions**
- → Hard to predict the behavior in the long run → **Deterministic *chaos***

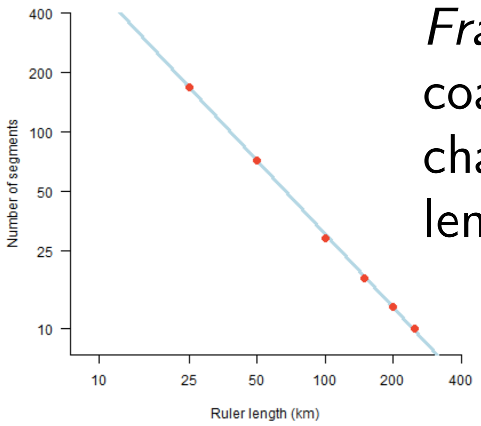
Stranger attractors, next …



Strange attractor







Fractal dimension: coastline length changes with the length of the ruler