15-382 Collective Intelligence – S18

Lecture 9:
Dynamical Systems 8

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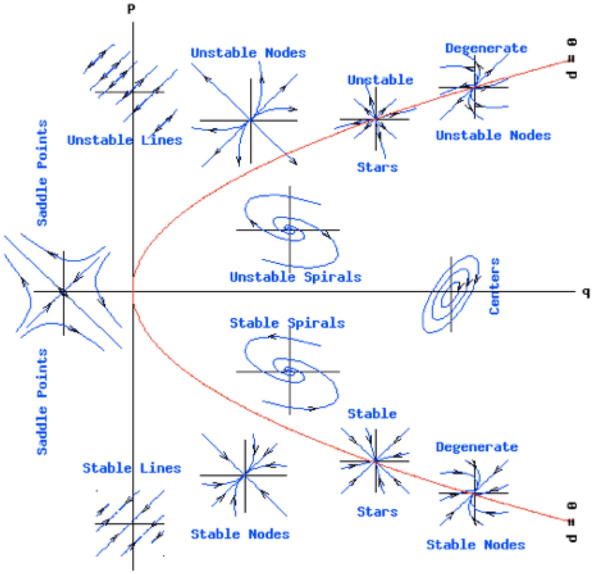
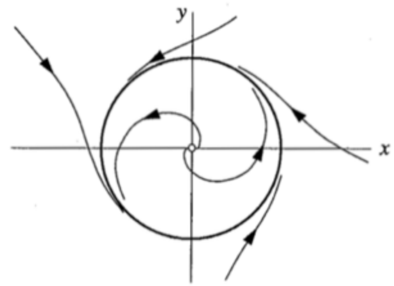
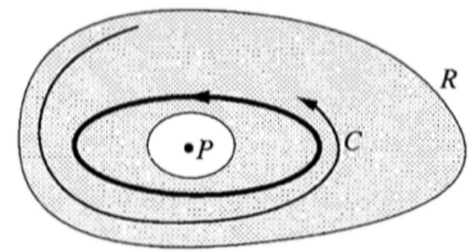
Fixed-point, Periodic, Strange attractors





Up to second-order systems, $n \leq 2$

Poincaré-Bendixson theorem



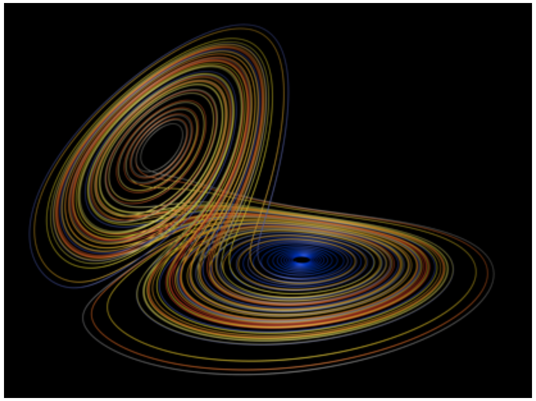
Regular attractors:

- Points (topological dimension: 0)
- Curves (topological dimension: 1)

For higher order systems, $n \geq 3$, novel geometry of attractors and complicated aperiodic dynamics can be observed

Strange attractors:

- Fractal dimension $\neq$ Topological dimension
- Lorenz attractor: Fractal dimension 2.06



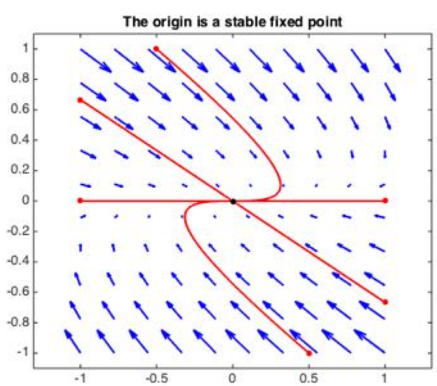
ATTRACTORS

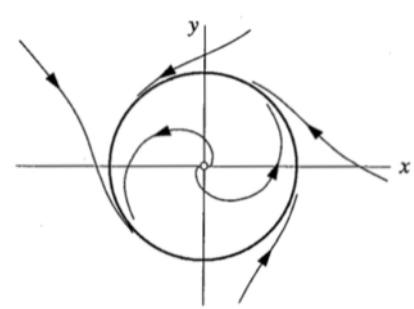
Informally: a set to which all neighboring trajectories converge

Attractor:

- A closed set A
- A is an invariant set: any trajectory $\boldsymbol{x}(t)$ that starts in A stays in A
- A attracts, as $t \to \infty$ an open set of initial conditions: there is an open set U that contains A, such that, if $\boldsymbol{x}(0) \in U$, $\boldsymbol{x}(t)$ tends to A as $t \to \infty$. A attracts an open set of initial conditions that starts near A. The largest set U is A's basin of attraction
- A is minimal: there's no proper subset of A that satisfies previous properties

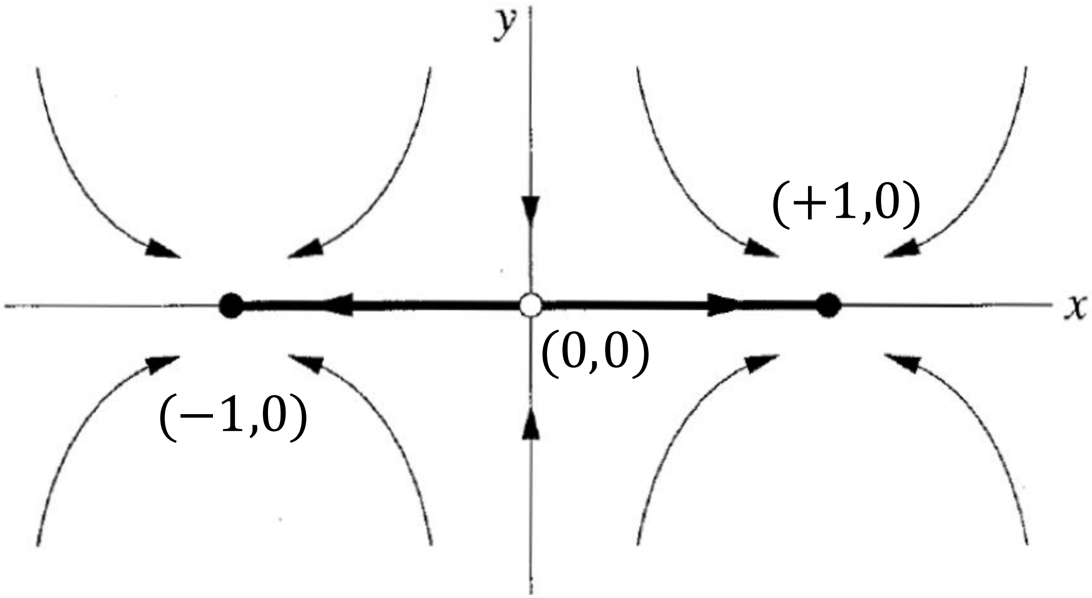
Stable fixed points





Stable limit cycles

EXAMPLE



Is $I = \{-1 \leq x \leq 1, y = 0\}$ an attractor?

✓ Closed set

✓ A is an invariant set

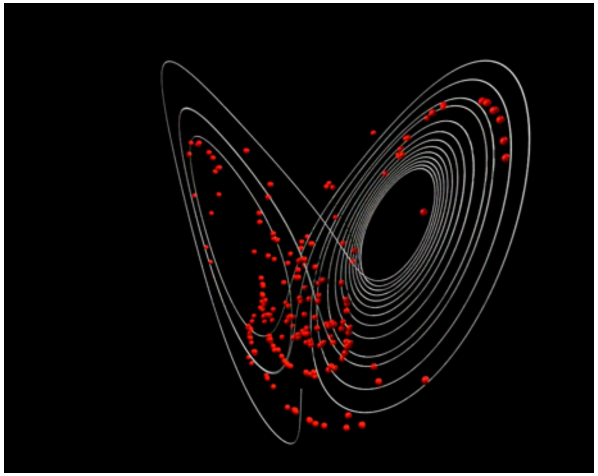
✓ As $t \to \infty$, it attracts an open set of initial conditions:

❖ Is minimal ☹. No, the fixed points ($\pm$1,0) are inside the closed set I. Actually they are the only attractors for the system

Strange Attractors

Strange Attractor:

- An attractor that exhibits sensitive dependence on initial conditions
  - Two initial conditions in the set U that are arbitrarily close at $t = 0$, become far significantly far apart as t grows over time, but still remain confined in the set that defines the attractor
- Geometrically: Has *fractal* dimension
- Deterministically chaotic attractors



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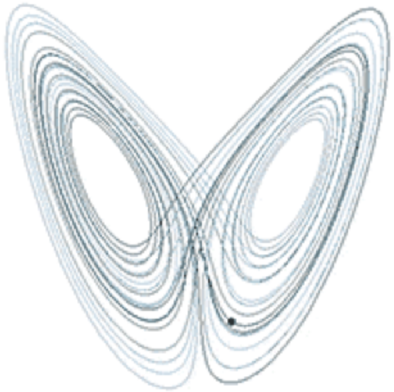
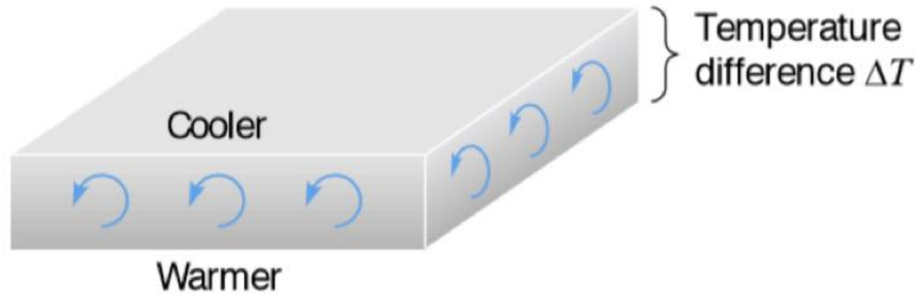


https://en.wikipedia.org/wiki/File:A_Trajectory_Through_Phase_Space_in_a_Lorenz_Attractor.gif

Lorenz system (1963)

$$
\begin{aligned}
dx/dt &= \sigma(-x + y), \\
dy/dt &= rx - y - xz, \\
dz/dt &= -bz + xy.
\end{aligned}
$$

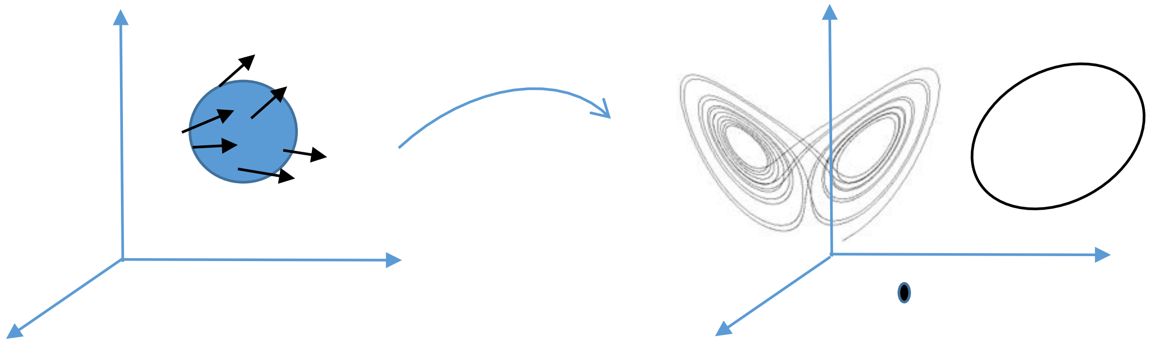


- x is related to the intensity of fluid motion from bottom to up
- y, z are related to temperature variations, respectively, horizontally and vertically
- σ, b are related to the material and geometrical properties of the fluid, and on earth's atmosphere is reasonable to set $\sigma = 10, b = 8/3$
- r is proportional to the temperature difference between the layers and it's the most "interesting" parameter

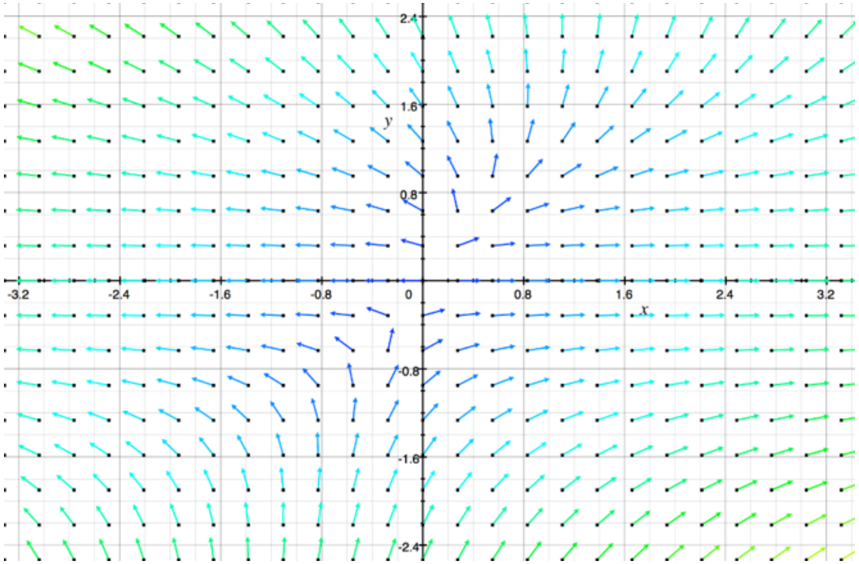
General properties

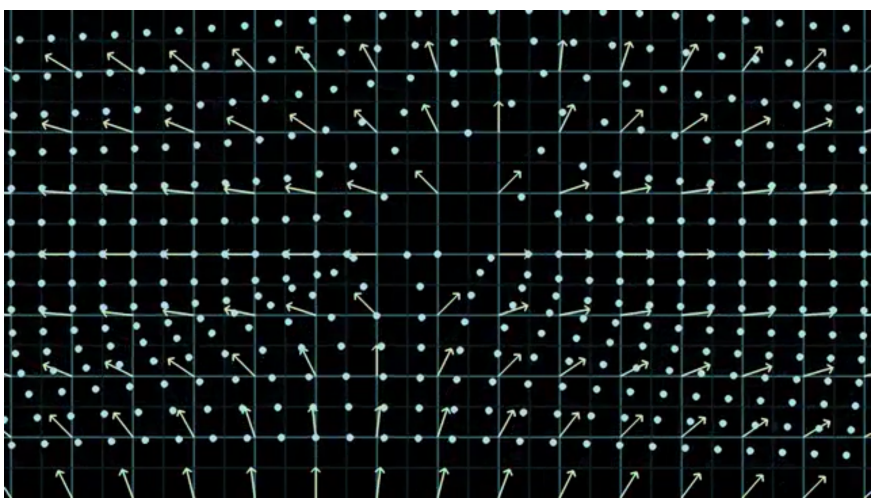
$$dx/dt = \sigma(-x + y),$$
$$dy/dt = rx - y - xz,$$
$$dz/dt = -bz + xy.$$

- **Symmetric in (x, y):** substituting (x, y) with $(-x, -y)$ doesn't change the system → if $\big(x(t), y(t), z(t)\big)$ is a solution → $\big(-x(t), -y(t), z(t)\big)$ is also a solution
- **Dissipative system**: volumes in the phase space contract under the flows



Flows and Divergence





A vector field $\boldsymbol{v}$

$$\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots$$

$$\nabla \cdot \vec{\mathbf{v}} = \overbrace{\frac{\partial v_1}{\partial x}}^{\text{Change in density in the } x\text{-direction}} + \underbrace{\frac{\partial v_2}{\partial y}}_{\text{Change in density in the } y\text{-direction}}$$

Divergence operator of the field $\boldsymbol{v}$ is a scalar function of $(x, y, \ldots)$ that measures the change in a point due to the vector field

Think of a vector field in terms of the *fluid flow* that it could generate: the change in density of particles about a point is measured by the *divergence*

Flows and Divergence

Components of the vector-valued function $\vec{\mathbf{v}}$

$$\overbrace{\nabla}^{\text{“divergence of } \vec{\mathbf{v}} \text{”}} \cdot \vec{\mathbf{v}}(x, y, \dots) = \frac{\partial v_1}{\partial x}(x, y, \dots) + \frac{\partial v_2}{\partial y}(x, y, \dots) + \cdots$$

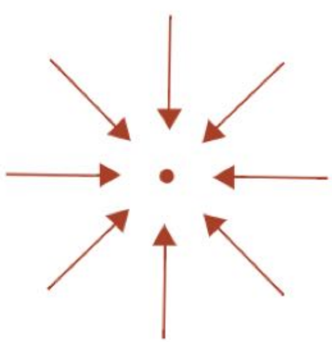
$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1(x, y, \dots) \\ v_2(x, y, \dots) \\ \vdots \end{bmatrix}}$$

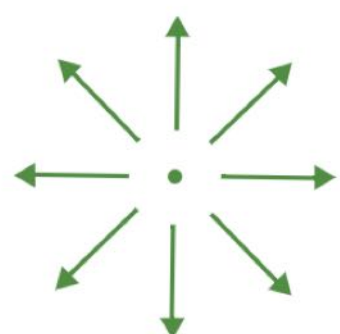
$$\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots$$

$$\nabla \cdot \vec{\mathbf{v}} = \overbrace{\frac{\partial v_1}{\partial x}}^{\substack{\text{Change in density} \\ \text{in the } x\text{-direction}}} + \underbrace{\frac{\partial v_2}{\partial y}}_{\substack{\text{Change in density} \\ \text{in the } y\text{-direction}}}$$

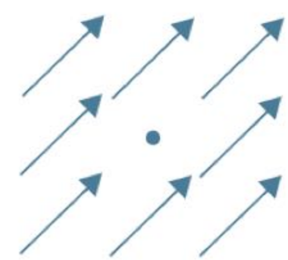
Density increase	Density decrease	Density unchanged
$\nabla \cdot \vec{\mathbf{v}} < 0$	$\nabla \cdot \vec{\mathbf{v}} > 0$	$\nabla \cdot \vec{\mathbf{v}} = 0$



Point is a sink



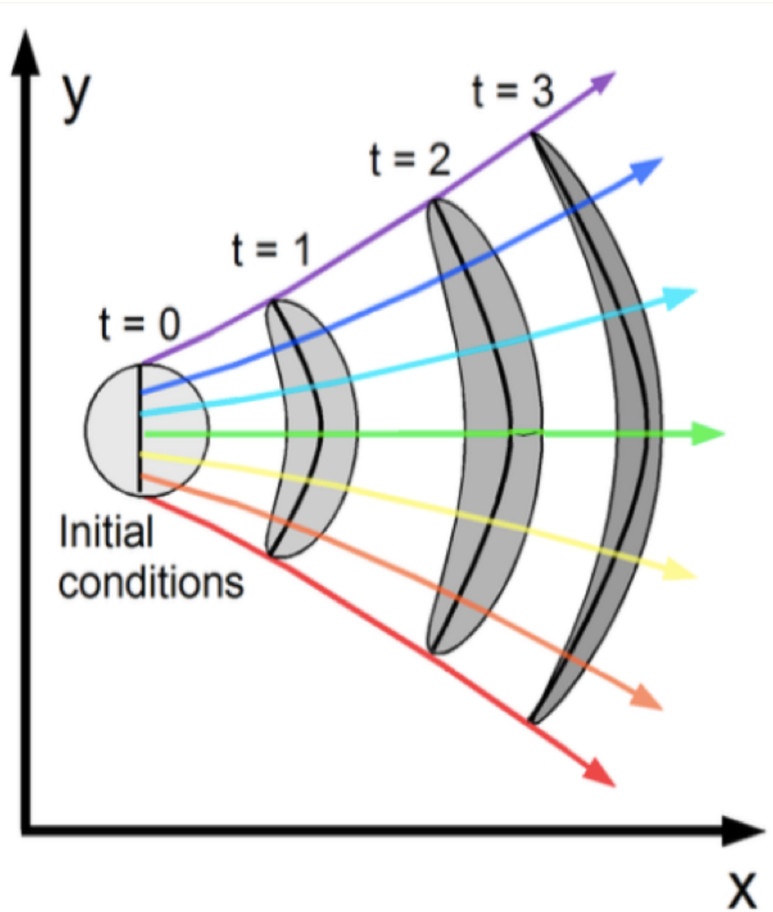
Point is a source



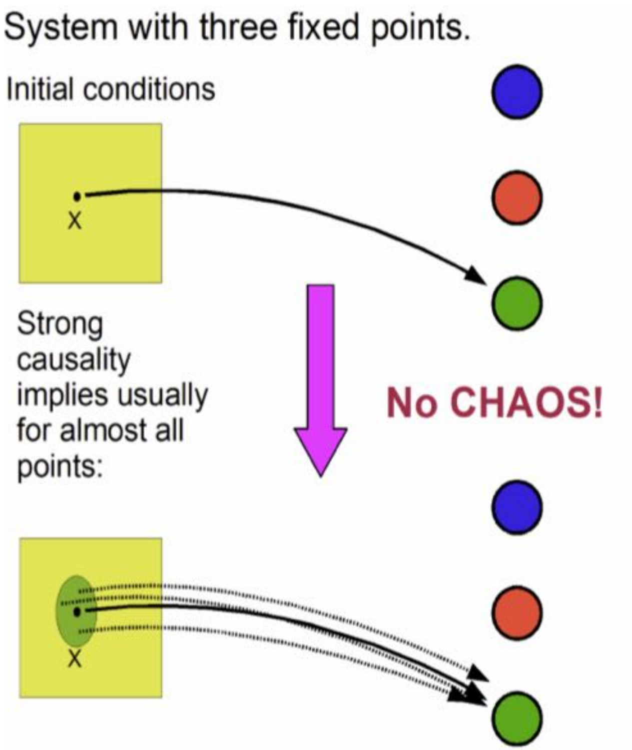
Point is a transit

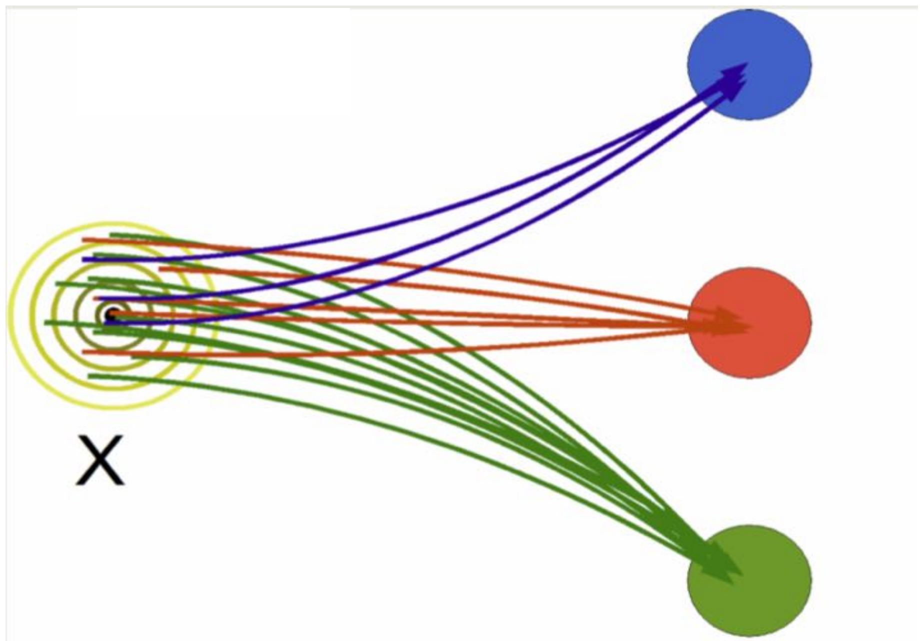
Flows of initial conditions in the phase space

How a solid ball of initial conditions gets transformed by the flows of the dynamical system? (think about the previous analogy with solid points, with infinitely many of them, all packed in an n-dimensional ball)



Regular vs. Chaotic scenario

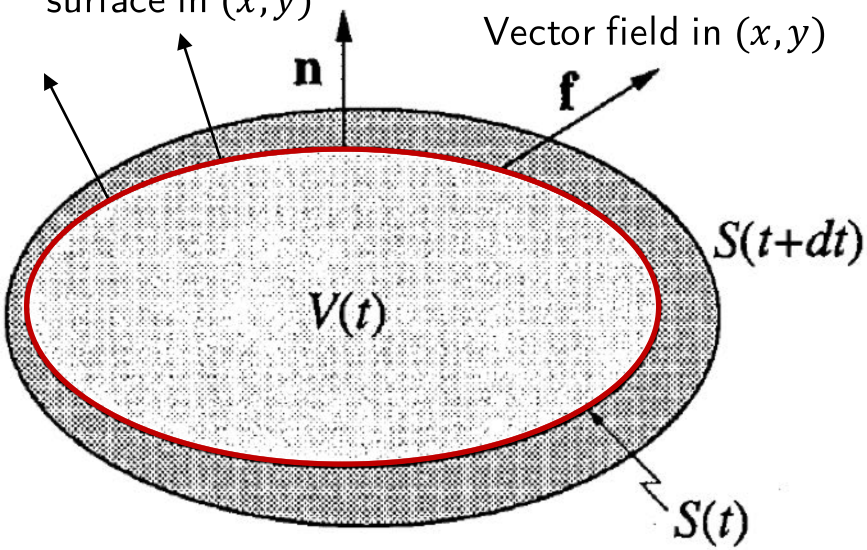




If point X goes to the green attractor, the same happens for all points in an open neighborhood about X. The volume of the initial conditions may stretch or contract but will not be dispersed, they will stick together

Point X goes to the green attractor, but the same does *not* happen for the points in its neighborhood. In the example, they end up in different attractors, but, more in general, they will end up generating different aperiodic orbits, *dispersing* the volume of the initial conditions

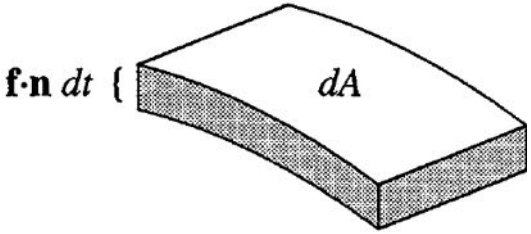
VOLUME CONTRACTION: FORMALIZATION



Side view of the volume

- Closed surface $S(t)$ of a volume $V(t)$ in the phase space
- (infinite) Set of initial conditions
- Let's evolve it for $dt \rightarrow S(t+dt)$
- What is the volume $V(t+dt)$?

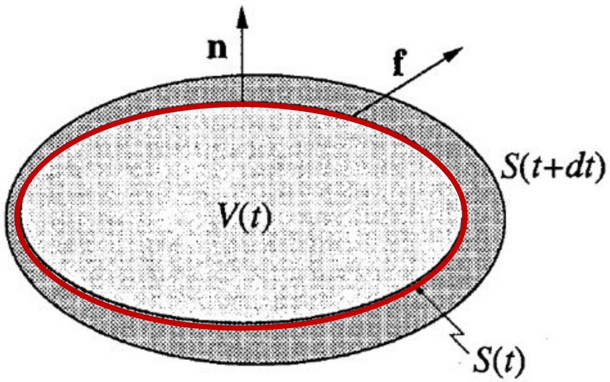
- $\boldsymbol{f}$ is the instantaneous velocity of the points subject to the field
- In dt, a patch of area dA, sweeps out a volume $(\boldsymbol{f} \cdot \boldsymbol{n}\, dt)dA$



$V(t+dt) = V(t)$ + (volume swept out by tiny patches of surface, integrated over all patches),

$$V(t+dt) = V(t) + \int_S (\mathbf{f} \cdot \mathbf{n}\, dt)\, dA$$

Volume contraction



$$V(t+dt) = V(t) + \int_S (\mathbf{f}\cdot\mathbf{n}\,dt)\,dA$$

$$\dot{V} = \frac{V(t+dt)-V(t)}{dt} = \int_S \mathbf{f}\cdot\mathbf{n}\;dA$$

Divergence theorem in 3D: the total flux across the boundaries of a surface S, that in our case is $\int_S \boldsymbol{f}\cdot\boldsymbol{n}\,dA$, equals the total divergence of the vector field $\boldsymbol{f}$ inside the entire volume V enclosed by the surface, $\int_V \nabla\cdot\boldsymbol{f}\,dV$

$$\Rightarrow \quad \dot{V} = \int_V \nabla\cdot\mathbf{f}\;dV$$

$$\nabla\cdot\mathbf{f} = \frac{\partial}{\partial x}[\sigma(y-x)] + \frac{\partial}{\partial y}[rx-y-xz] + \frac{\partial}{\partial z}[xy-bz]$$

Lorenz system

$$= -\sigma - 1 - b < 0. \qquad V(t) = V(0)e^{-(\sigma+1+b)t}.$$

$$\dot{V} = -(\sigma+1+b)V \qquad V(t) = V(0)e^{-(\sigma+1+b)t}$$

Volumes shrink exponential fast!

No repelling

- A Lorenz system cannot have repelling fixed points or repelling closed orbits
  - Repellers are in contradiction for volume contraction, since they are sources of volumes
  - Let's enclose a repeller with a solid surface of initial conditions nearby in the phase space
  - A short time later, the surface (e.g. a sphere) will have expanded because the trajectories are driven away
  - → Volume of the surface would increase and not decrease!

→ All fixed points must be sinks, or saddles

→ All closed orbits (if exists) must be stable or saddle-like

Fixed points

$$
\begin{aligned}
dx/dt &= \sigma(-x + y), \\
dy/dt &= rx - y - xz, \\
dz/dt &= -bz + xy.
\end{aligned}
$$

$$
\begin{aligned}
x(r - 1 - z) &= 0, \\
-bz + x^2 &= 0.
\end{aligned}
$$

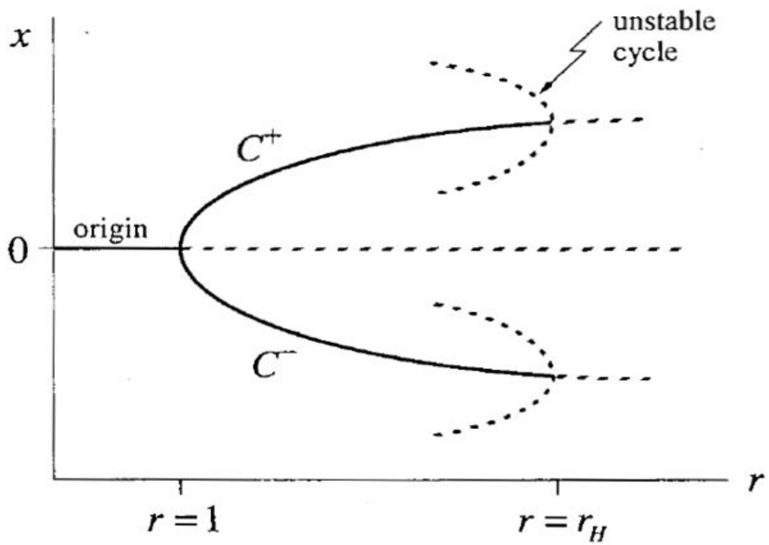
(0,0,0) is a critical point for all values of r, asymptotically stable for $r < 1$

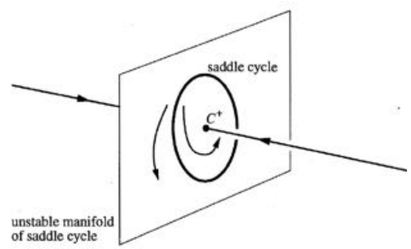
C^+ $\left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$

C^- $\left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$

Additional critical points for $r > 1$, linearly stable for

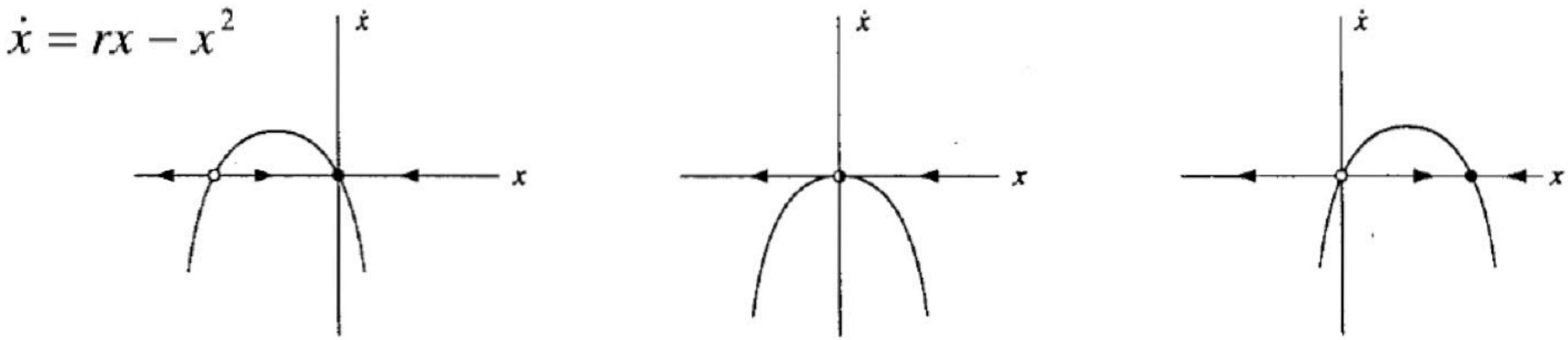
$$1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$





For $r > r_H$ everything seems to be unstable and diverge....

Bifurcations



(a) $r<0$ (b) $r=0$ (c) $r>0$

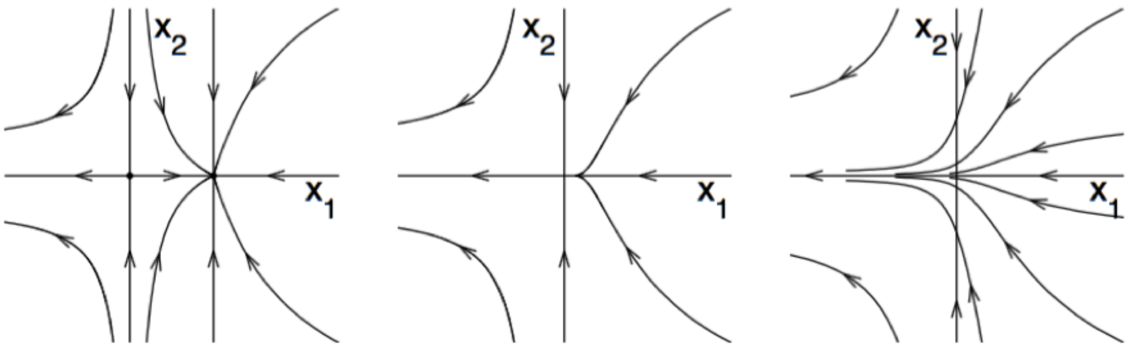
- Continuously changing the parameter r, determines changes in both the number of critical points and in their stability
- A sort of *mechanics* (i.e. motions and forces) seems to arise in the phase space, determining attractions, collisions, transfers of properties, and generation of new critical points out of old ones → **Bifurcations**
- From $r < 0 \rightarrow r = 0$, the unstable saddle point "moves" toward the stable node in (0,0), and at $r = 0$ they *collide:* the resulting new critical point is half-stable (~ it inherits the properties of both critical points)
- As soon as r turns to a positive values, a new critical point, a stable node, appears, while the previous, half-stable point at the origin, remains a critical point but becomes unstable for all $r > 0$

Saddle-node bifurcation

$$\dot{x}_1 = \mu - x_1^2, \qquad \dot{x}_2 = -x_2$$

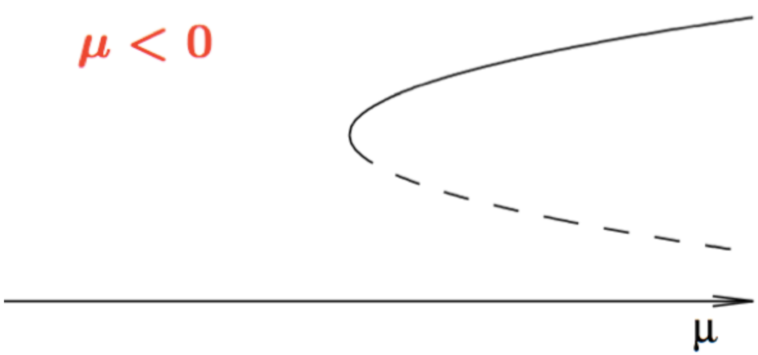
No equilibrium points when $\mu < 0$

As μ decreases, the saddle and node approach each other, collide at $\mu = 0$, and disappear for $\mu < 0$



$\mu > 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu < 0$

Bifurcation diagram



Saddle-node bifurcation

Dangerous / hard bifurcation

The change remove critical points

Transcritical bifurcation

$$\dot{x}_1 = \mu x_1 - x_1^2, \qquad \dot{x}_2 = -x_2$$

Two critical points: (0,0) and (μ, 0)

The Jacobian at $(0, 0)$ is $\begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}$

(0,0) is a stable node for $\mu < 0$,
a saddle for $\mu > 0$

The Jacobian at $(\mu, 0)$ is $\begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}$

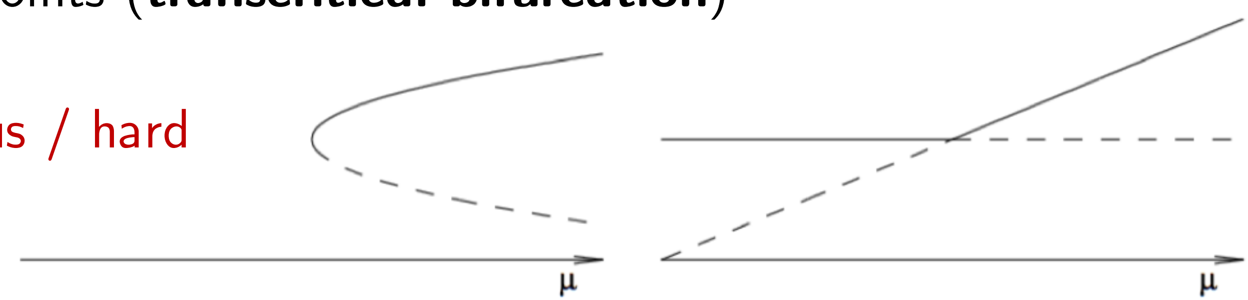
(μ,0) is a saddle for $\mu < 0$,
a stable node for $\mu > 0$

An eigenvalue crosses the origin as μ crosses zero

While the equilibrium point persists through the bifurcation at $\mu = 0$, the point (0,0) changes from a stable node to a saddle, and the point (μ, 0) changes from a saddle to a stable node: they *swap their stability*, without changing the number of critical points (**transcritical bifurcation**)

Dangerous / hard

Safe / soft



(a) Saddle–node bifurcation (b) Transcritical bifurcation

PITCHFORK BIFURCATION: SUPERCRITICAL

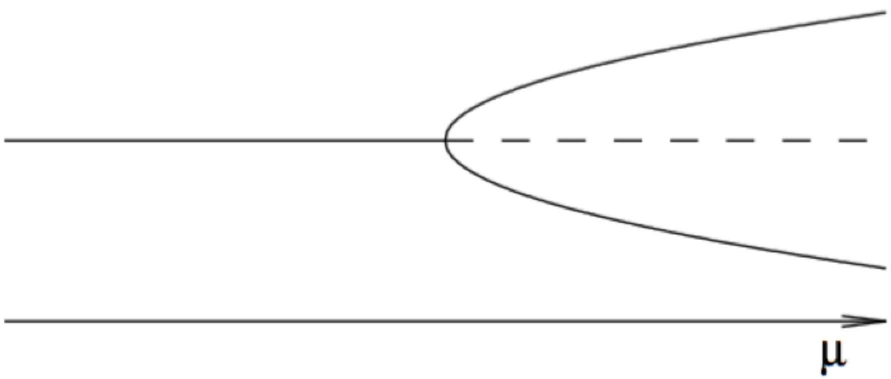
$$\dot{x}_1 = \mu x_1 - x_1^3, \qquad \dot{x}_2 = -x_2$$

For $\mu < 0$, there is a stable node at the origin

For $\mu > 0$, there are three equilibrium points: a saddle at $(0, 0)$ and stable nodes at $(\sqrt{\mu}, 0)$, and $(-\sqrt{\mu}, 0)$

Safe / soft

New equilibrium points, stable are generated



Pitchfork bifurcation - Supercritical

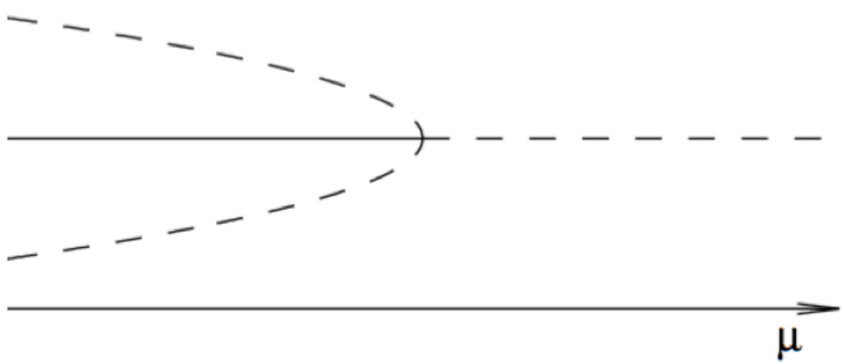
Two new stable equilibrium are generated at the bifurcation. The original, equilibrium point, *changes its stability*, from stable to unstable

$$\dot{x}_1 = \mu x_1 + x_1^3, \qquad \dot{x}_2 = -x_2$$

For $\mu < 0$, there are three equilibrium points: a stable node at $(0,0)$ and two saddles at $(\pm\sqrt{-\mu}, 0)$

For $\mu > 0$, there is a saddle at $(0,0)$

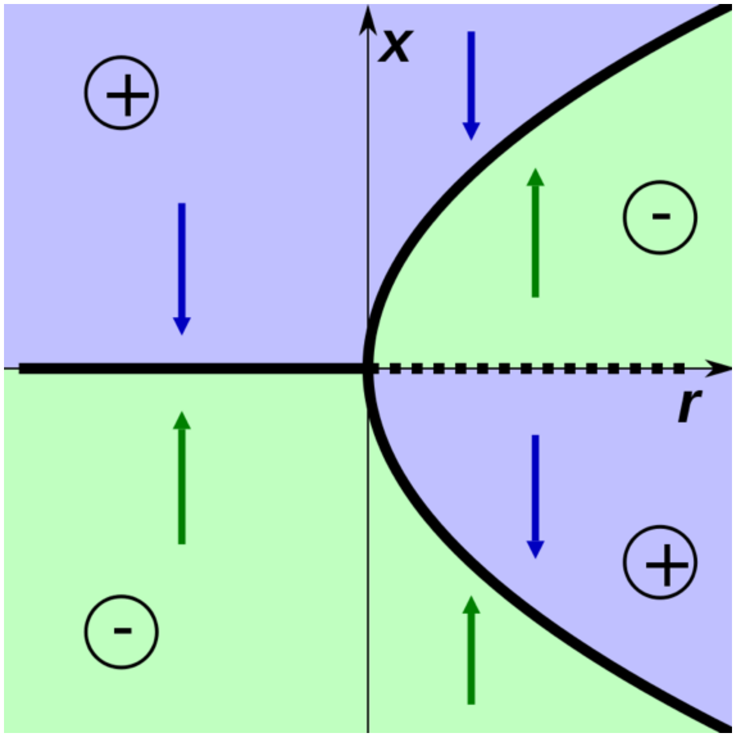
Dangerous / hard
Two unstable equilibrium points are absorbed in the previously unstable one, that in turn loses its stability



Pitchfork bifurcation - Subcritical

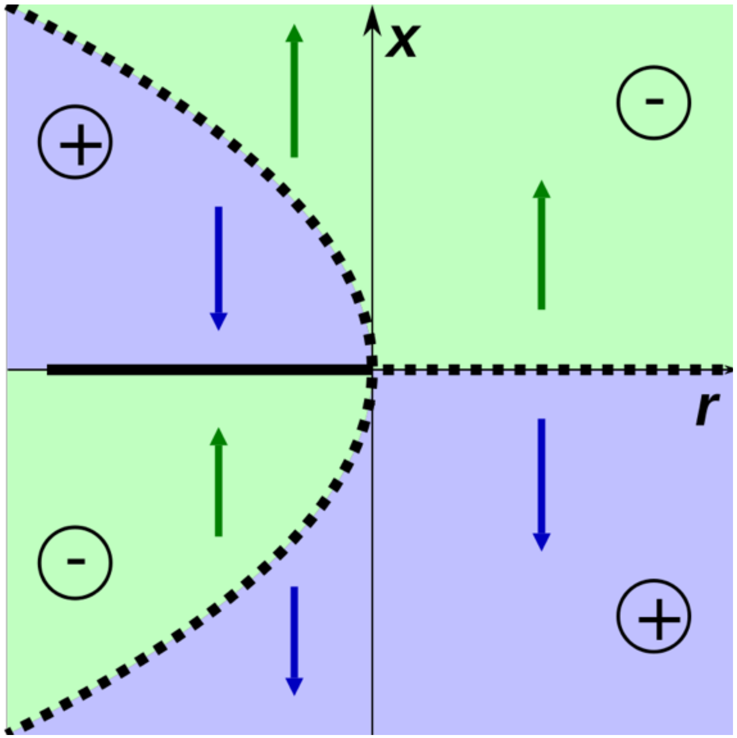
Pitchfork bifurcation

Supercritical / Safe



$$\dot{x} = rx - x^3$$

Subcritical / Dangerous



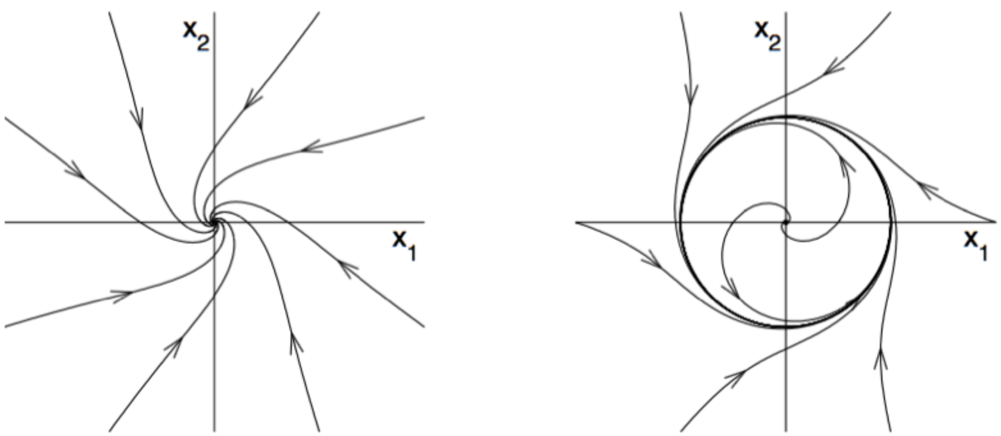
$$\dot{x} = rx + x^3$$

Hopf bifurcation

- A point where a system's stability switches and a periodic solution arises
- It is a local bifurcation in which, depending on a parameter, a fixed point loses stability, as a pair of complex conjugate eigenvalues (of the linearization around the fixed point) crosses the complex plane imaginary axis. Under reasonably generic assumptions about the dynamical system, a small-amplitude limit cycle branches from the fixed point.

$$\dot{r} = \mu r - r^3 \quad \text{and} \quad \dot{\theta} = 1$$

For $\mu > 0$, there is a stable limit cycle at $r = \sqrt{\mu}$

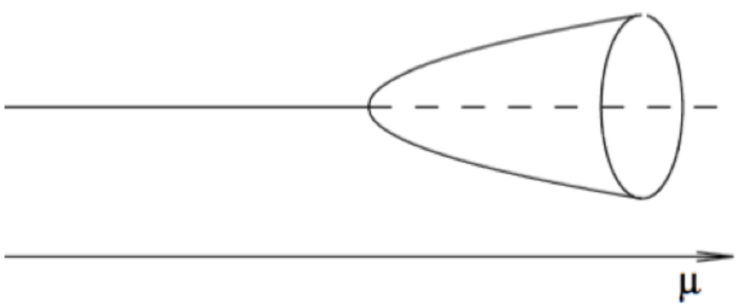


$\mu < 0$ $\mu > 0$

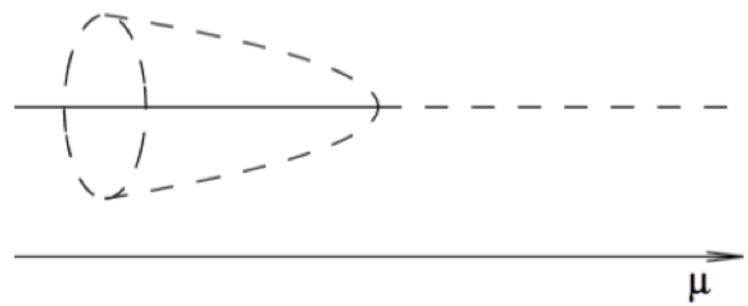
HOPF BIFURCATION

As μ increases from negative to positive values, the stable focus at the origin merges with the unstable limit cycle and bifurcates into unstable focus

Subcritical Hopf bifurcation



(e) Supercritical Hopf bifurcation

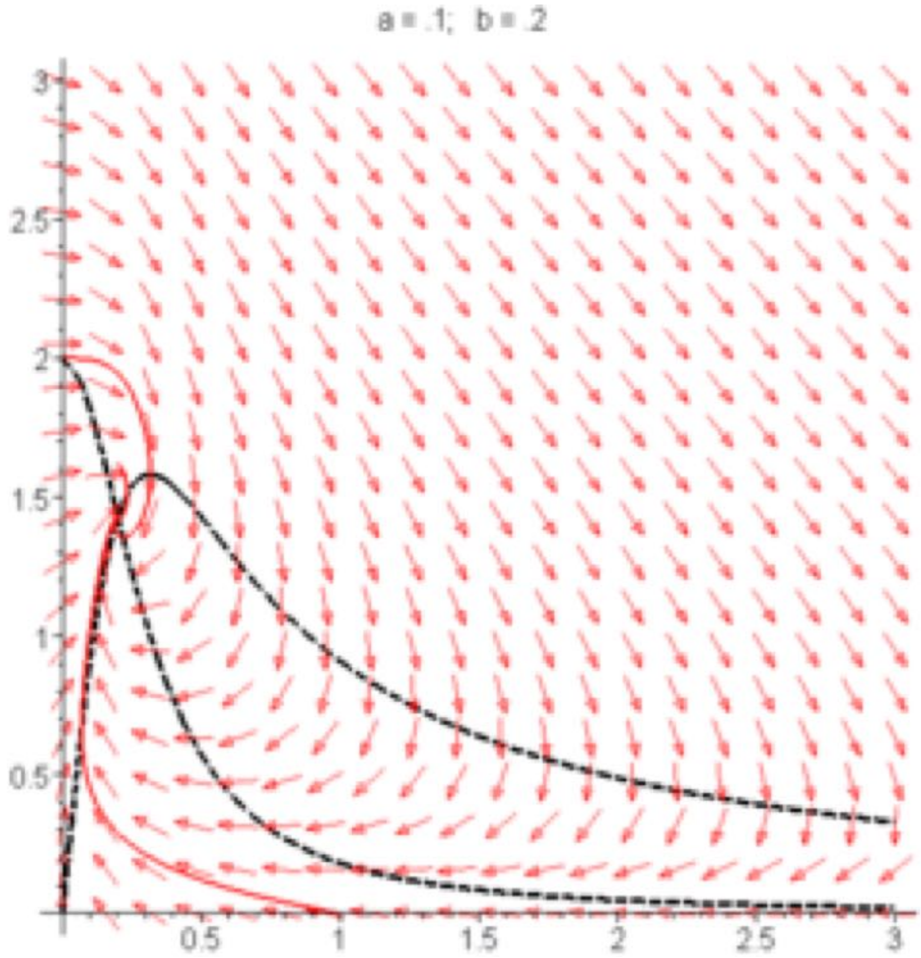


(f) Subcrtitical Hopf bifurcation

safe or soft

dangerous or hard

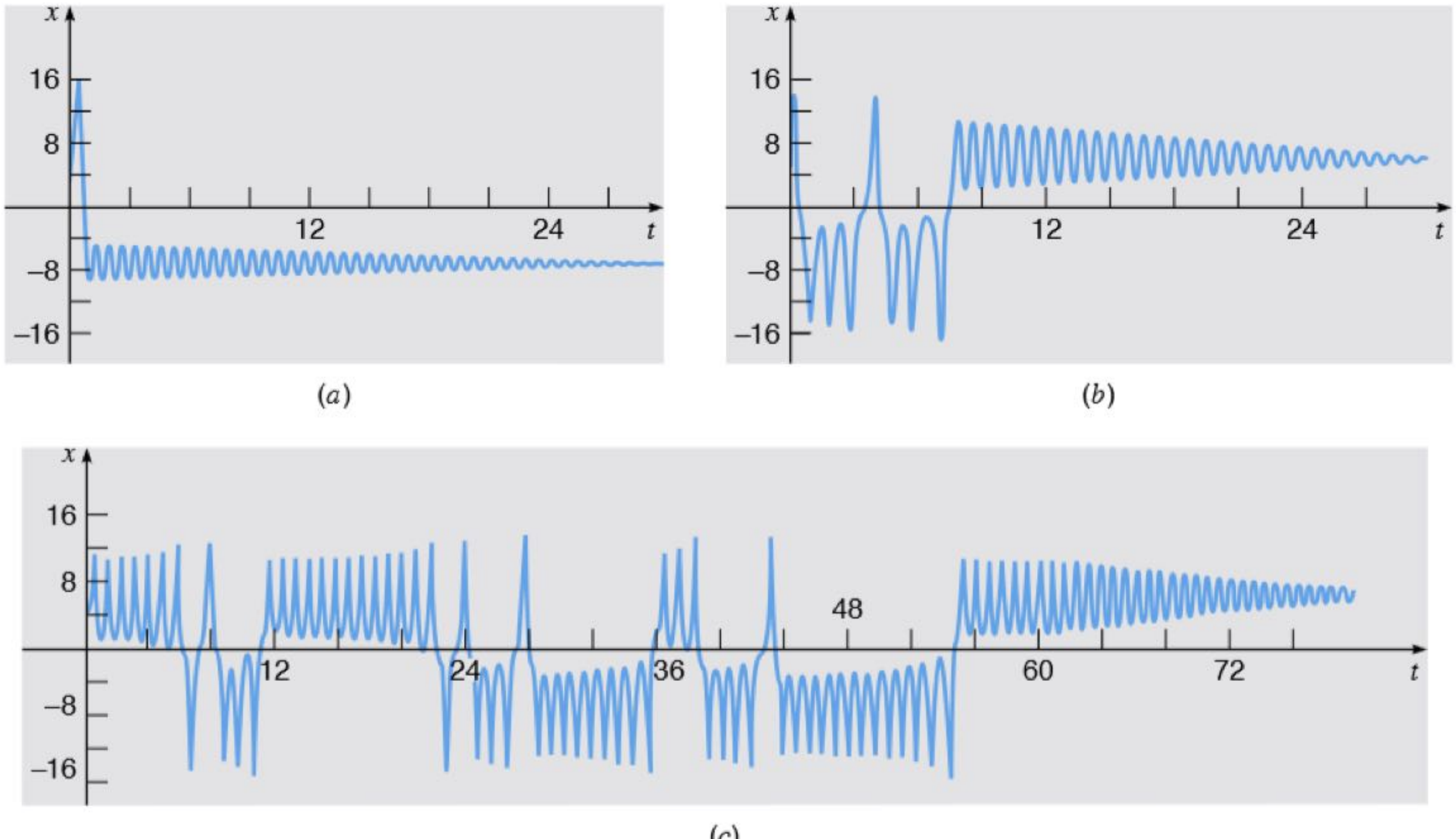
Hopf bifurcation



https://en.wikipedia.org/wiki/File:Hopf-bif.gif

Let's go back to our Lorenz system ....

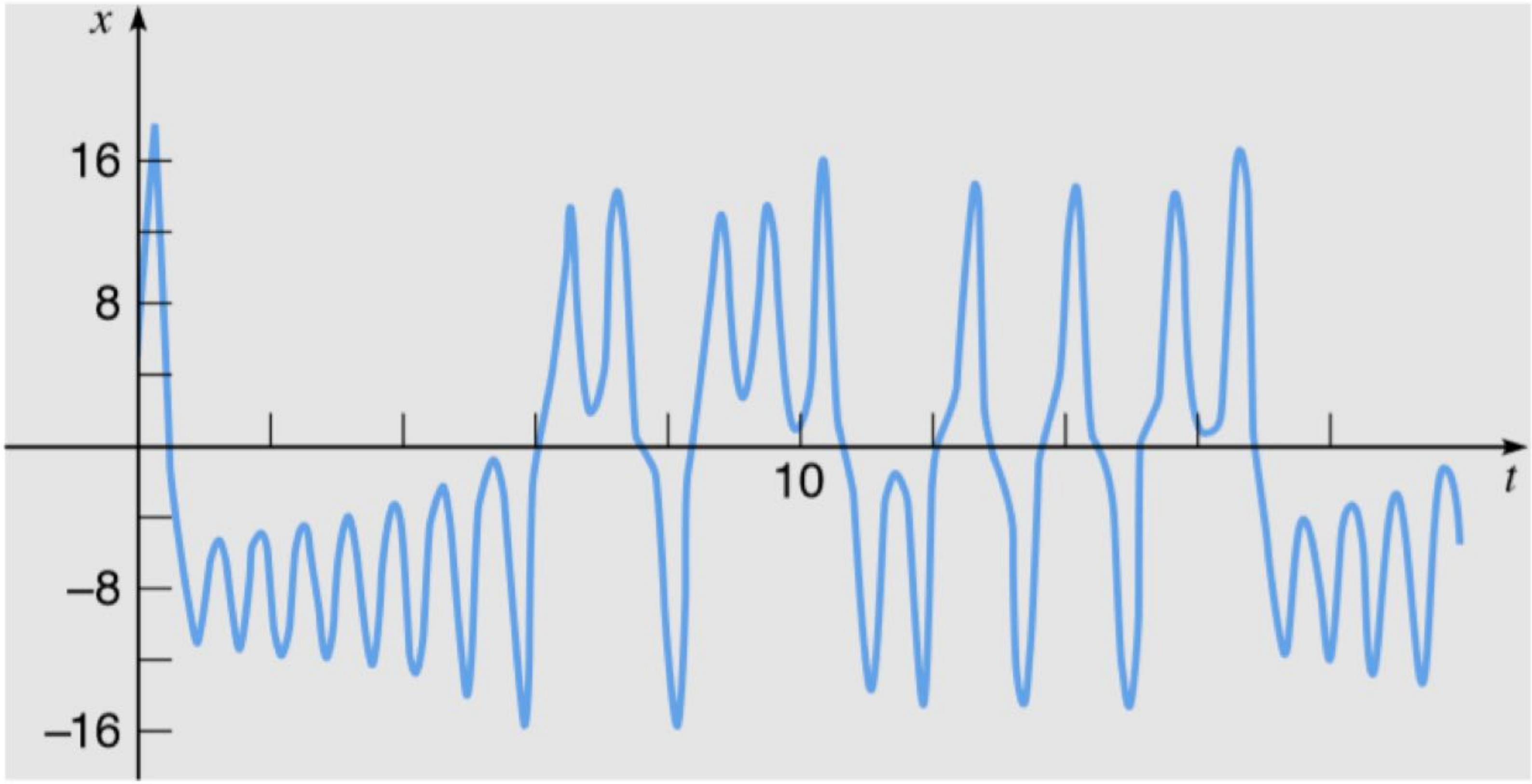
$r = 21$, different initial conditions



The convergence gets much longer, depending on the initial condition

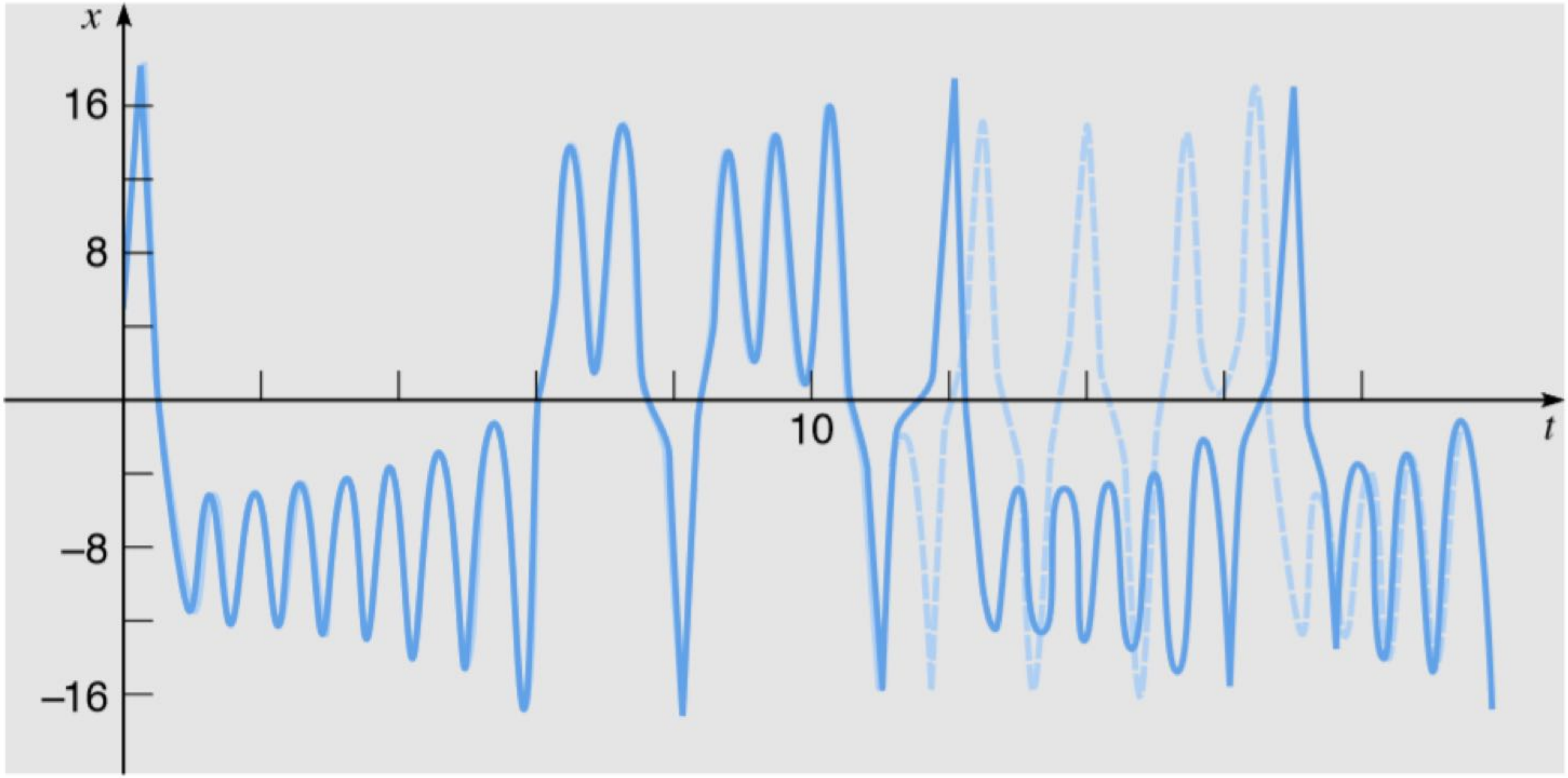
What happens for larger r?

$r = 28$



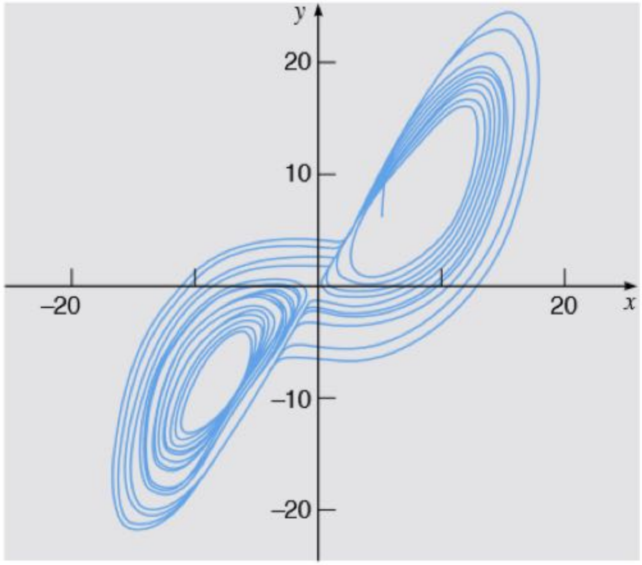
Aperiodic behavior, the values are however confined in [-16,16]

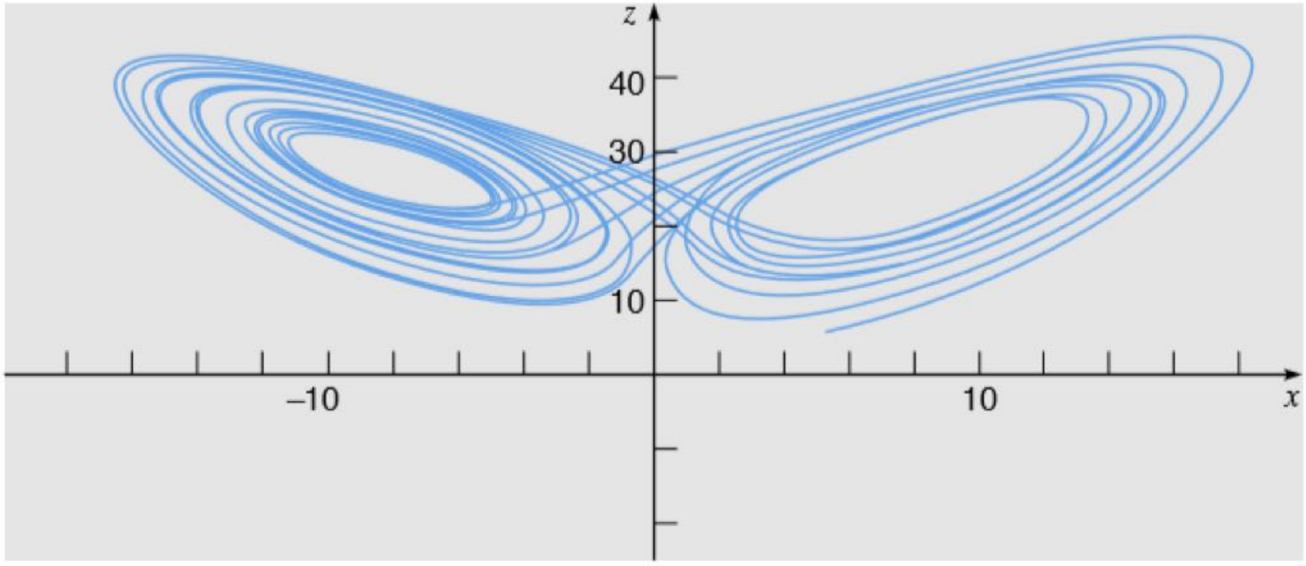
Two Similar initial conditions



They start together but they completely diverge from each other, still being bounded in the excursion of the values

In the phase space





Try it yourself!

- Check the *python functions* and *animations* in the file `viz-attractor.py` on course website

```
def plot_trajectories(dynamical_sys, initial_conditions, colors, seconds, time_step):

def show_two_dimensional_projections(x, y, z):

def show_individual_variables(x, y, z, t):

def set_point_on_trajectory(i, states, states_near, attractor):

def move_trajectory_point(dynamical_sys, initial_condition, seconds, time_step):

def lorenz(state, t):

def rossler(state, t):

def lotka_volterra_3d(state, t):

....

if __name__ == '__main__':
  # Set of (arbitrary) four initial conditions, that will be plotted as four trajectories
  # in different colors. One of the trajectories is also plotted in the two dimensional planes (xy, xz, yz)
  # and will be used for animate the motion on the attractor.
  # Initial conditions and evolution times are provided for the three included examples non-linear systems
  #
  initial_conditions_lorenz = [[1.0, 1.0, 1.0], [3.0, 3.0, 3.0], [-10, -10, 10], [-5,-2,0]]
  seconds_lorenz = 40.0
  time_step_lorenz = 0.01

  initial_conditions_rossler = [[10, -10, 5], [5, -20, 10], [5, 5, 5], [10,-2,10]]
  seconds_rossler = 200.0
  time_step_rossler = 0.01

  initial_conditions_lv = [[0.25, 0.5, 2.5], [0.5, 0.5, 2], [1, 0.5, 1.5], [1.7, 0.5, 1.2]]
  seconds_lv = 200.0
  time_step_lv = 0.01


  # select the system to show and get its parameters
  dynamical_system = lorenz
  initial_conditions = initial_conditions_lorenz
  seconds = seconds_lorenz
  time_step = time_step_lorenz

  colors = ["r", "g", "b", "y"]


  plot_trajectories(dynamical_system, initial_conditions, colors, seconds, time_step)

  move_trajectory_point(dynamical_system, initial_conditions[0], seconds, time_step)

plt.close('all')
```

DETERMINISTIC CHAOS, A DEFINITION

No definition of the term *chaos* is universally accepted yet, but almost everyone would agree on the three ingredients used in the following working definition:

> ***Chaos*** is *aperiodic long-term behavior* in a *deterministic* system that exhibits *sensitive dependence on initial conditions*.

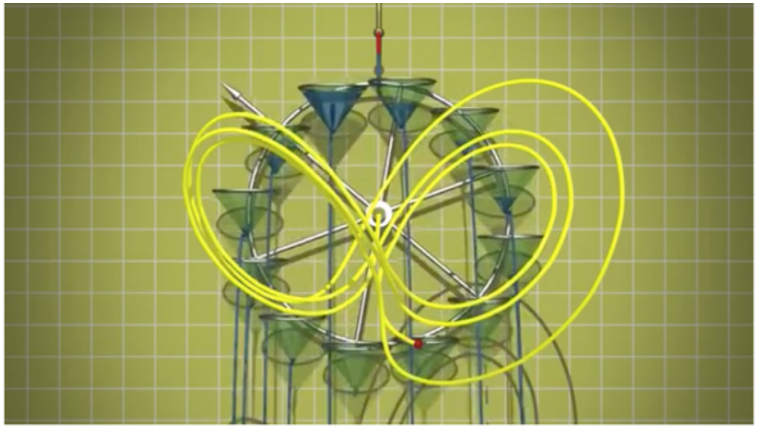
1. "Aperiodic long-term behavior" means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$. For practical reasons, we should require that such trajectories are not too rare. For instance, we could insist that there be an open set of initial conditions leading to aperiodic trajectories, or perhaps that such trajectories should occur with nonzero probability, given

2. "Deterministic" means that the system has no random or noisy inputs or parameters. The irregular behavior arises from the system's nonlinearity, rather than from noisy driving forces.
3. "Sensitive dependence on initial conditions" means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent.

Videos to watch!

- Two beautifully instructive videos about deterministic chaos:

https://www.youtube.com/watch?v=c0gDLEHbYCk



https://www.youtube.com/watch?v=SlwEt5QhAGY