15-382 COLLECTIVE INTELLIGENCE – S18

LECTURE 10:
DISCRETE-TIME DYNAMICAL SYSTEMS 1

INSTRUCTOR:
GIANNI A. DI CARO

جامعة كارنيجي ميلون في قطر
Carnegie Mellon University Qatar

(More) General definition of dynamical systems

A dynamical system is a 3-tuple (T, S, Φ):

- S is a set of all possible **states** of the dynamical system (the state space)
- T is the set of values the **time (evolution) parameter** can take
- Φ is the **evolution function** of the dynamical system, that associates to each $\boldsymbol{x} \in S$ a unique image in S depending on the time parameter t, (not all pairs $(t, \boldsymbol{x})$ are feasible, that requires the subset D)

$$\Phi: D \subseteq T \times S \to S$$

  - $\Phi(0, \boldsymbol{x}) = \boldsymbol{x}$ (the initial condition)
  - $\Phi\big(t_2, \Phi(t_1, \boldsymbol{x})\big) = \Phi(t_2 + t_1, \boldsymbol{x})$, (property of states)

    for $t_1, t_1 + t_2 \in I(x)$, $t_2 \in I(\Phi(t_1 x))$,

    $I(x) = \{t \in T : (t, \boldsymbol{x}) \in D\}$
  - The evolution function Φ provides the system state (the *value*) at time t for any initial state $\boldsymbol{x}$
  - $\gamma_x = \{\Phi(t, \boldsymbol{x}) : t \in I(\boldsymbol{x})\}$ **orbit** of the system through $\boldsymbol{x}$, starting in $\boldsymbol{x}$, the set of visited states as a function of time

Types of Dynamical systems

- Informally: A dynamical system defines a *deterministic rule* that allows to know the current *state* as a function of *past states*
- Given an initial condition $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in S$, a deterministic trajectory $\boldsymbol{x}(t),\ t \in I(\boldsymbol{x}_0)$ is produced by (T, S, Φ)
- States can be "anything" mathematically well-behaved that represent situations of interest
- **Continuous time dynamical systems (Flows):** T open interval of $\mathbb{R}$, Φ continous and differentiable function → *Differential* equations

Ordinary Differential Equations

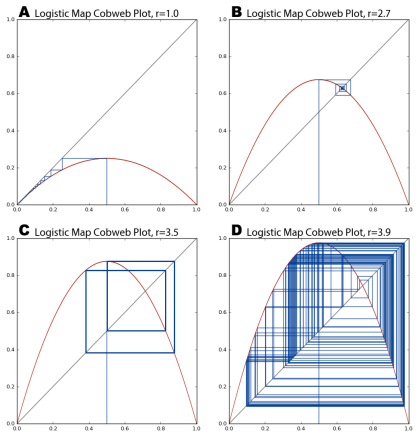
$$\begin{cases} \frac{dx_1}{dt} = x_1(g_1 - c_1 x_1) \\ \frac{dx_2}{dt} = x_2(g_2 - c_2 x_2) \end{cases}$$

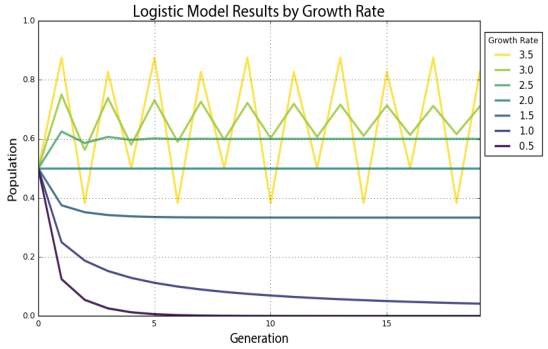
- $\dot{x} = f(x(t-\tau))$ *Delay models*
- $\dot{x} = f\big(x(t)\big) + \int_{t_0}^{t} f\big(x(\tau)\big)d\tau$ *Integro-Differential Equations*, accounting for history
- $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(x,t) = \frac{\partial^2}{\partial x^2}\varphi(x,t)$ *Partial Differential Equations*, accounting for space and time

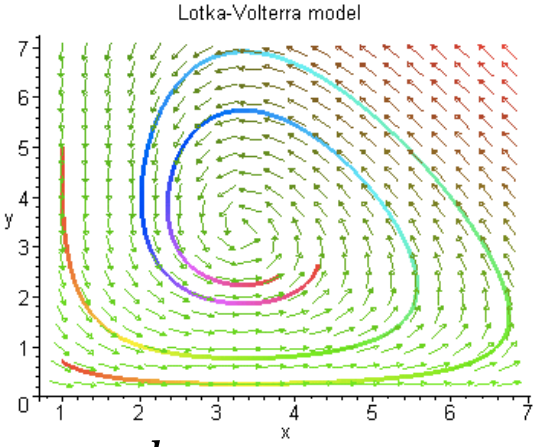
- **Discrete-time dynamical systems (Maps):** T interval of $\mathbb{Z}$, Φ a function

$$x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-m}),\ \text{iterated updating}$$

From local rules to global behaviors?







Maps $x_n = f(x_{n-1}, x_{n-2}, \dots, x_{n-m})$

$\Delta t = 1$, when $\Delta t \to 0$ → dt → Differential eq.

$$\frac{d\boldsymbol{x}}{dt} = f(\boldsymbol{x}, t)$$ Flows

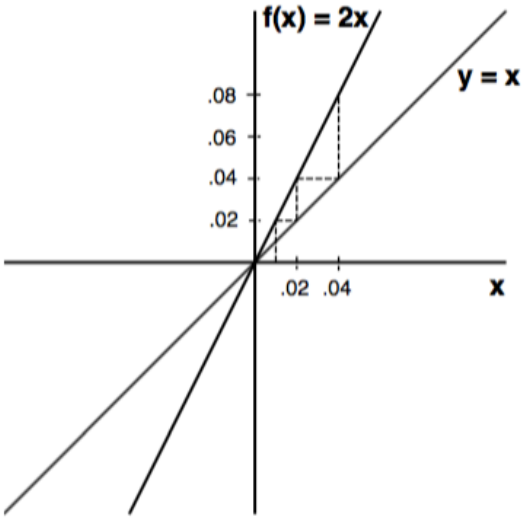
- For an infinitesimal time, only the instantaneous variation, the *velocity*, makes sense → The *next state* is expressed implicitly, and all the instantaneous variations, local in time, must be integrated in order to obtain the **global behavior** $\boldsymbol{x}(t)$
- Also in maps, the time-local iteration rule is a local description that can give rise to extremely complex global behaviors
- → How do we integrate the local description into global behaviors?
- → How do we predict global behaviors from the local descriptions?

Maps

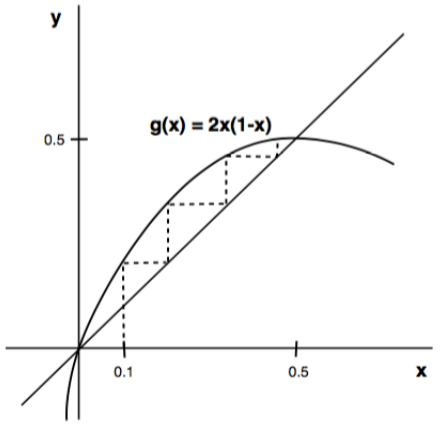
- Where maps can arise from?
  - *Inherently discrete-time processes:* looking at populations in terms of generations, epidemics in terms of weeks, economy in terms of quarters or years, traffic models per hour, growth per days, ...
  - *Discretization of differential equations*:
    - Euler method: $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \rightarrow \quad \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + h\boldsymbol{f}(\boldsymbol{x}_n)$, ...Runge-Kutta,...
  - *Discretization of algebraic equations:*
    - Newton's method for solving $f(x) = 0$ → Expand in Taylor series near x_n: $f(x) = f(x_n) + (x - x_n)f'(x_n) + \cdots$ taking the usual linear approximation: $f(x) \approx f(x_n) + (x - x_n)f'(x_n)$, equating to 0: $x_{n+1} = x_n - f(x_n)/f'(x_n)$
- Let's focus on **one-dimensional maps**
- Even in one dimension, iterated maps can produce incredibly complex behaviors, including deterministic chaos!
- Later on, we will consider *multi-dimensional maps* defined over a *lattice* (spatial grid) → **Cellular Automata** → even more complex behaviors

Maps: same terminology as in flows

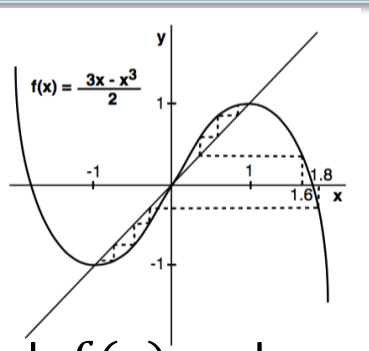
- $f(x) = 2x$, f is a map
- The **orbit** of x under the map f is the set of points: $\{x, f(x), f\big(f(x)\big), f\Big(f\big(f(x)\big)\Big), \dots\} = \{x, f(x), f^2(x), f^3(x), \dots\}$ corresponding to the iterated application of the map
- The initial point provides the **initial conditions**
- A point x^*, such that $f(x^*) = x^*$ is a **fixed point,** the orbits remain in x^* for all future iterations



Cobweb plots for individual orbits



Fixed points



- Starts at 1.6, converges to 1
- Starts at 1.8, converges to -1

Stability of a fixed point x^*?

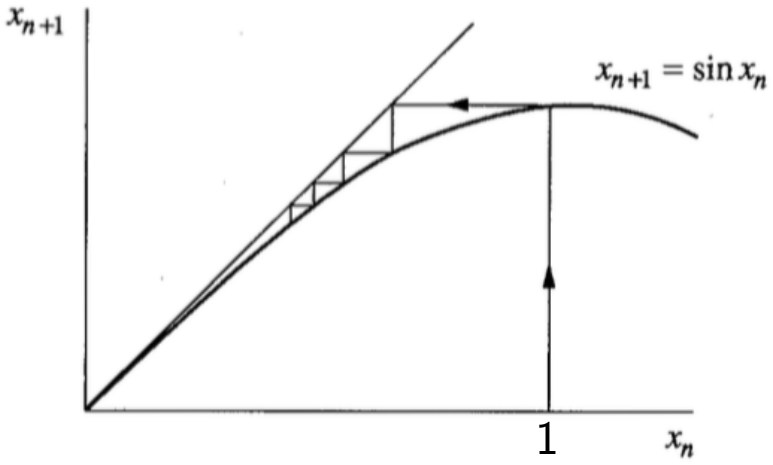
- Fixed points correspond to intersection between graph $f(x)$ and x
- General map: $x_{n+1} = f(x_n), \quad f(x^*) = x^*$
- Let's consider a near orbit, $x_n = x^* + \varepsilon_n$: is the orbit attracted or repelled from x^*? If it's attracted we can say that x^* is stable
- Does the perturbation ε_n grow or decay with n?
- By the definition, $x^* + \varepsilon_{n+1} = x_{n+1} = f(x^* + \varepsilon_n)$, and using the Taylor series expansion about x^*:

$$x^* + \varepsilon_{n+1} = x_{n+1} = f(x^* + \varepsilon_n) = f(x^*) + f'(x^*)\varepsilon_n + O({\varepsilon_n}^2)$$

- Given that $f(x^*) = x^* \rightarrow \varepsilon_{n+1} = f'(x^*)\varepsilon_n + O({\varepsilon_n}^2)$
- If we take the **linear approximation →**
- **Linearized map:** $\varepsilon_{n+1} = f'(x^*)\varepsilon_n$
- **Eigenvalue / multiplier:** $\lambda = f'(x^*)$

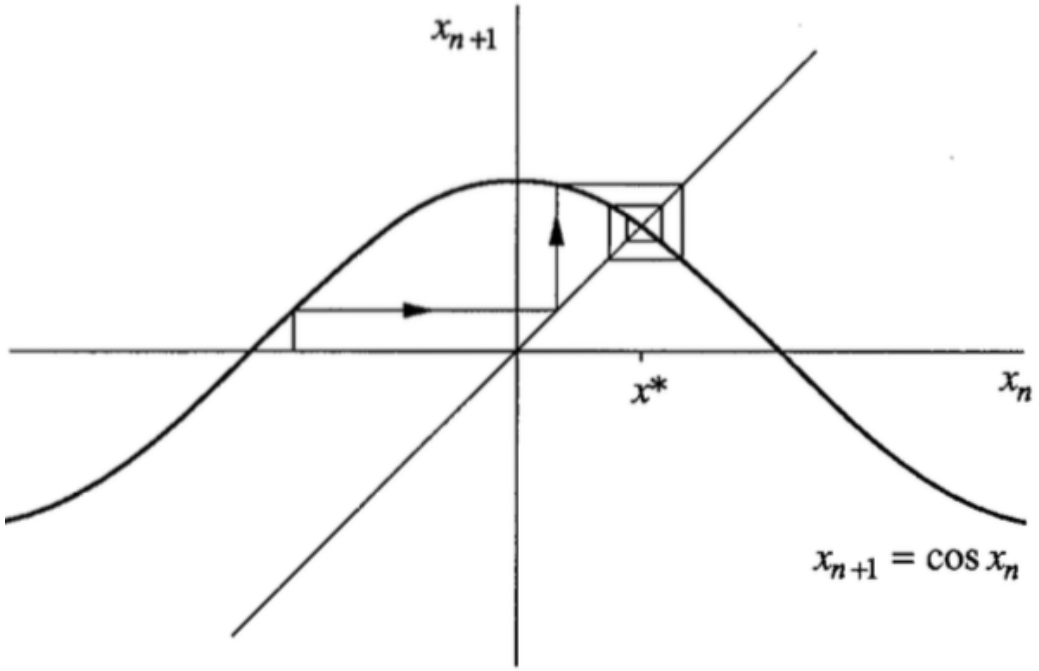
Fixed points

- Linearized map: $\varepsilon_{n+1} = f'(x^*)\varepsilon_n$
- Solution of linearized map: $\varepsilon_1 = \lambda\varepsilon_0,\ \varepsilon_2 = \lambda\varepsilon_1 = \lambda^2\varepsilon_0$ ..... $\varepsilon_n = \lambda^n\varepsilon_0$
- If $|\lambda| = |f'(x^*)| < 1$ → $\varepsilon_n \to 0$, for $n \to \infty$, and x^* is **linearly stable**
- If $|\lambda| = |f'(x^*)| > 1$ → $\varepsilon_n \to \infty$, for $n \to \infty$, and x^* is **unstable**
- The linear stability holds also for the general map
- The marginal case $|\lambda| = |f'(x^*)| = 1$ doesn't allow to draw conclusions. In this case the quadratic term $O(\varepsilon_n{}^2)$ determines the stability
- If $|\lambda|=0$, then the fixed point is said **superstable**



- $x_{n+1} = \sin x_n$
- $x^* = 0$ is a fixed point
- $\lambda = f'(x^*) = 1$, marginal case
- Cobweb → It's stable!
- Is it global? For *all* orbits $x_n \to 0$?
- For any x_0, $x_1 \in [-1,1]$ since $|\sin x_1| < 1$
- → From cobweb we can say it's global

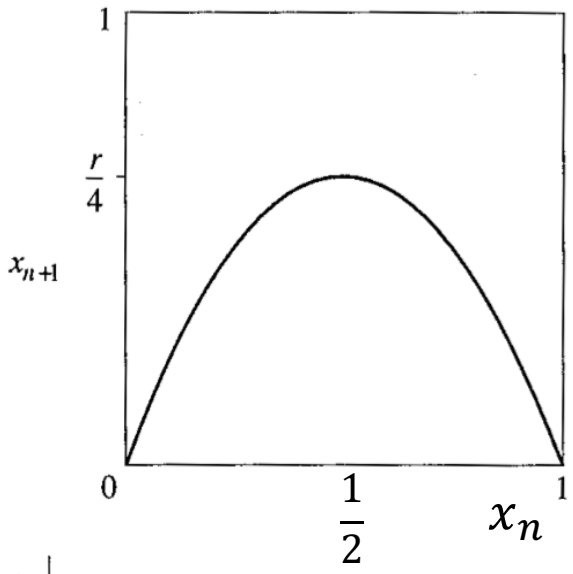
Another example, limiting behavior

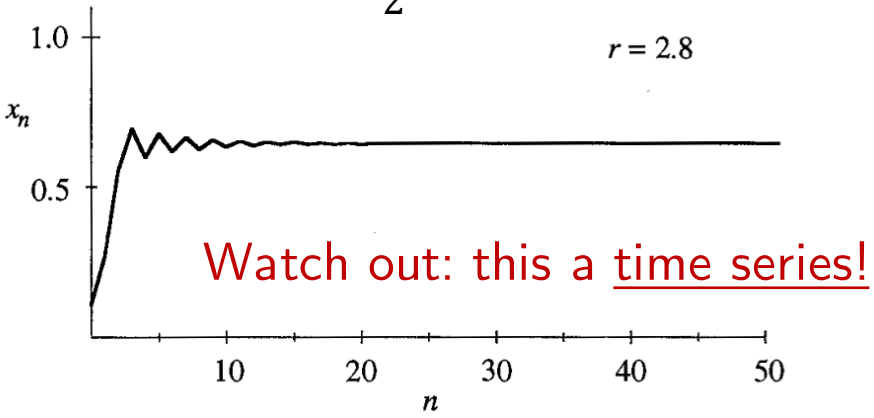


- $x_{n+1} = \cos x_n$
- $\lim_{n\to\infty} x_n$? ... by iterating the map (e.g., use calculator!), $x_n \to 0.739..$
- Solution of trascendental equation $x = \cos x$
- The fixed point 0.739... has $\lambda < 0$ → **Damped oscillations**
- For $0 < \lambda < 1$ convergence to a stable fixed point is **monotonic**

Logistic map

- $x_{n+1} = r x_n (1 - x_n)$
- x_n is a dimensionless measure of the population in the nth generation and r is the intrinsic growth rate (with capacity being limited to 1)





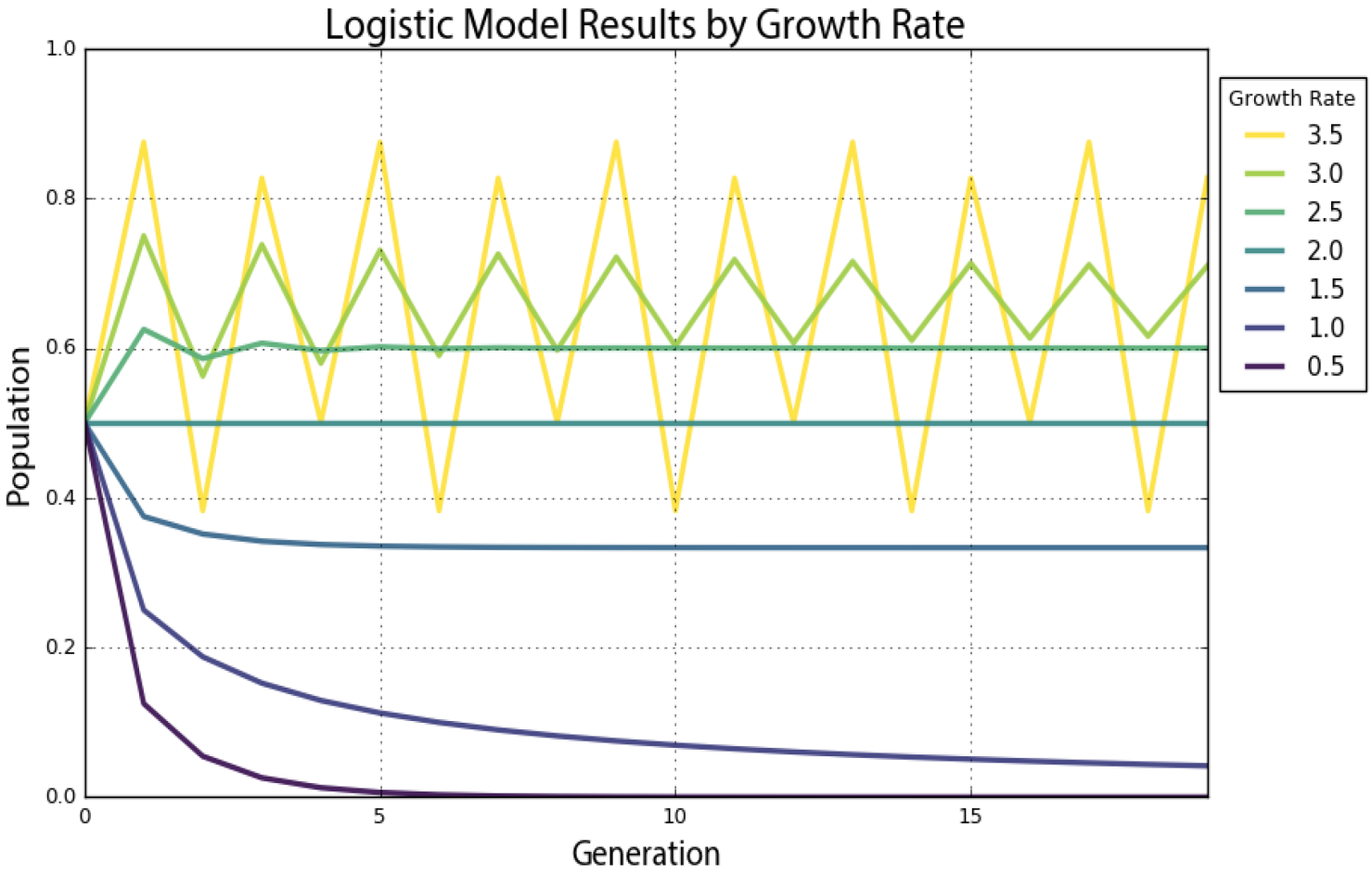
Watch out: this a time series!

- Let's restrict $0 \leq r \leq 4$

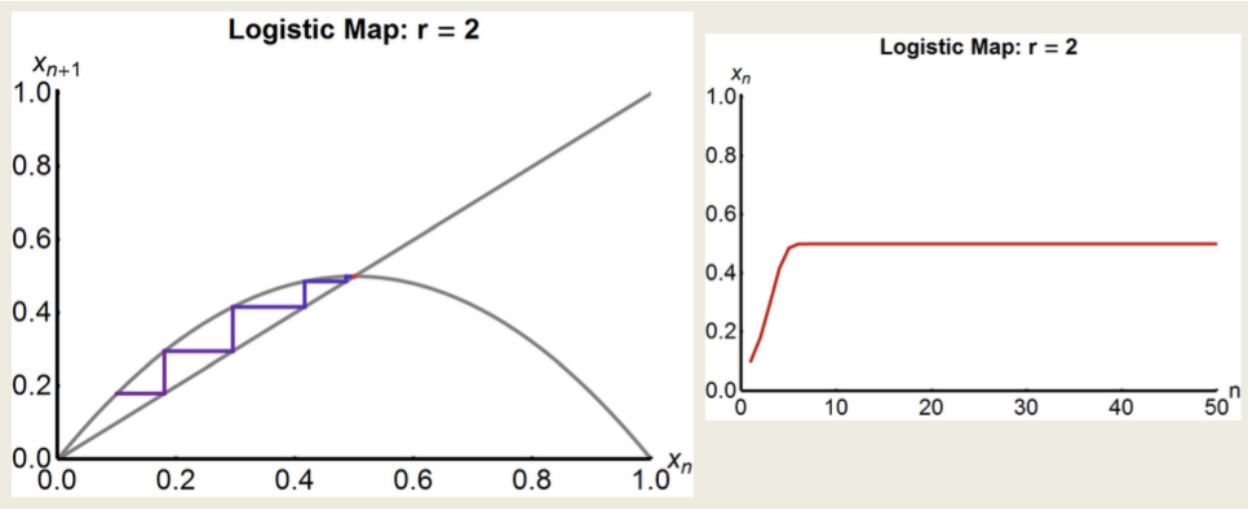
→ The map maps $[0,1] \rightarrow [0,1]$

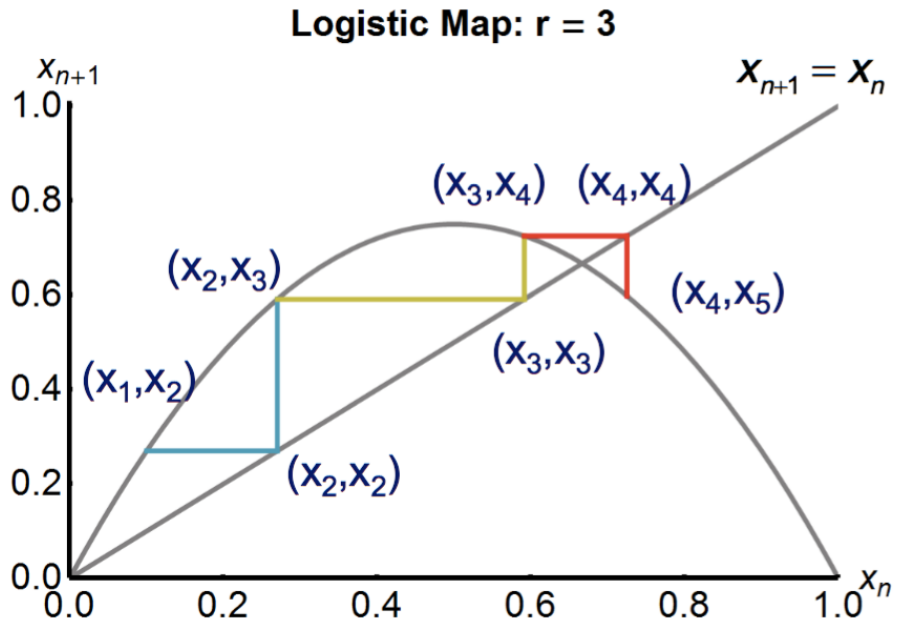
- Let's fix r and study the evolution
- Trivially, for small growth rates, $r < 1$, the population always goes extinct, as $x_n \rightarrow 0$
- For $1 < r < 3$, population grows and eventually reaches a non-zero steady state

A path to the chaos ...

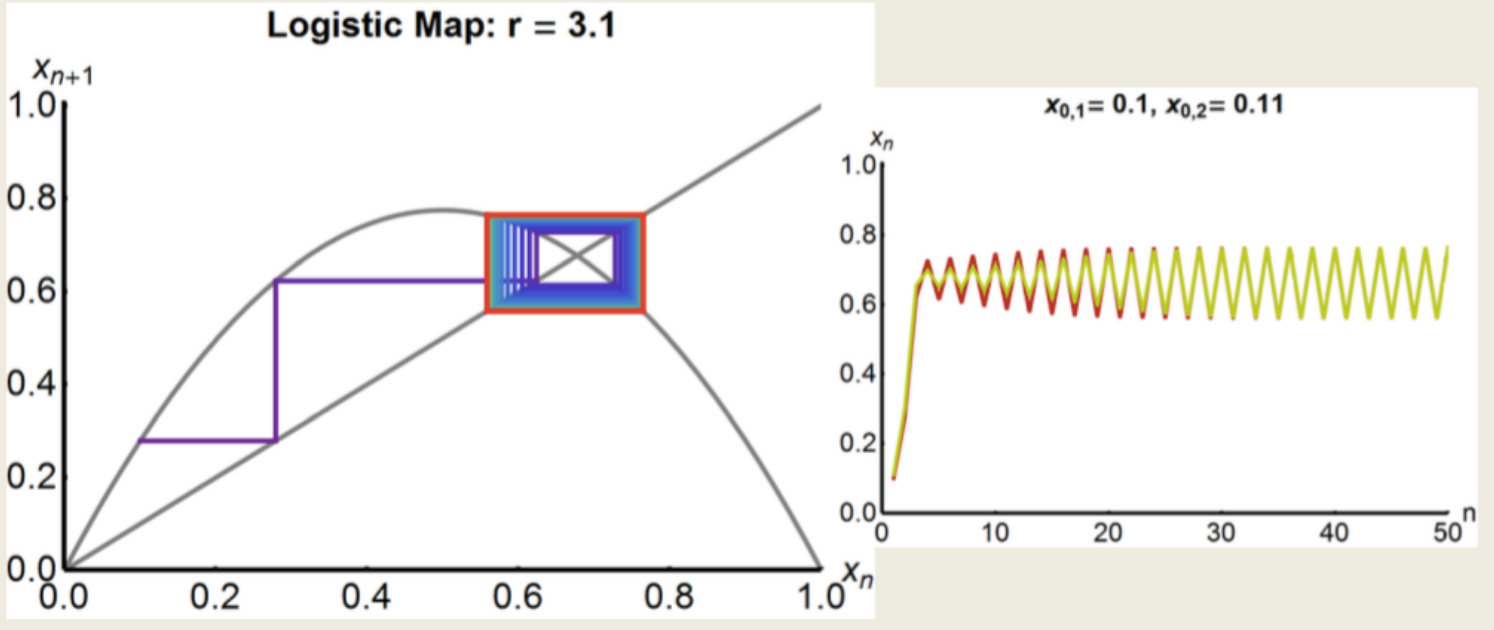


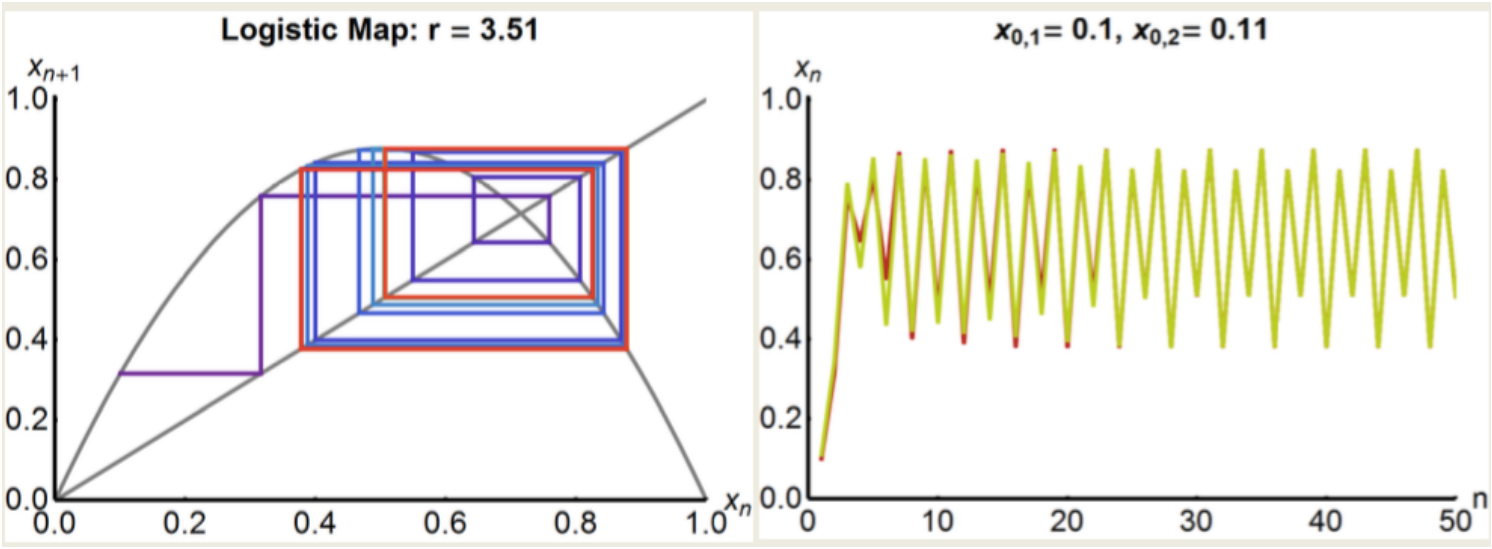
"Regular" behavior, periodic attractors



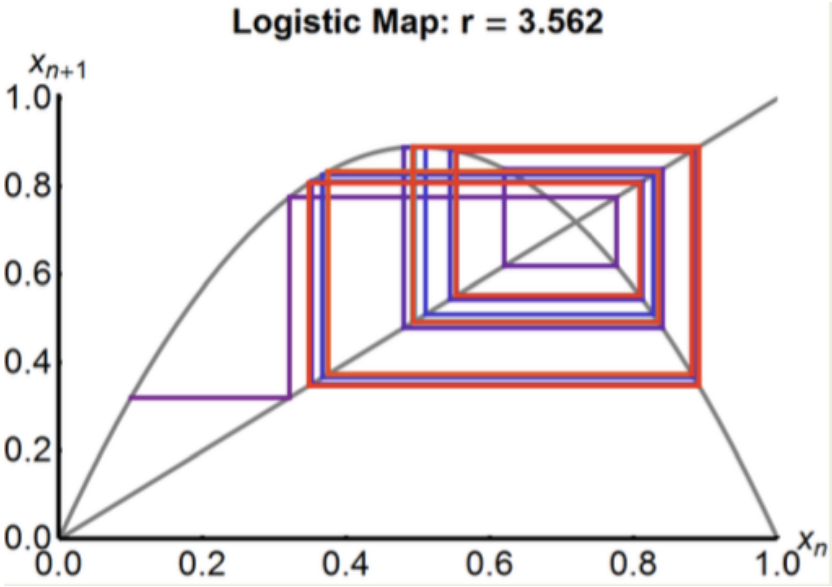


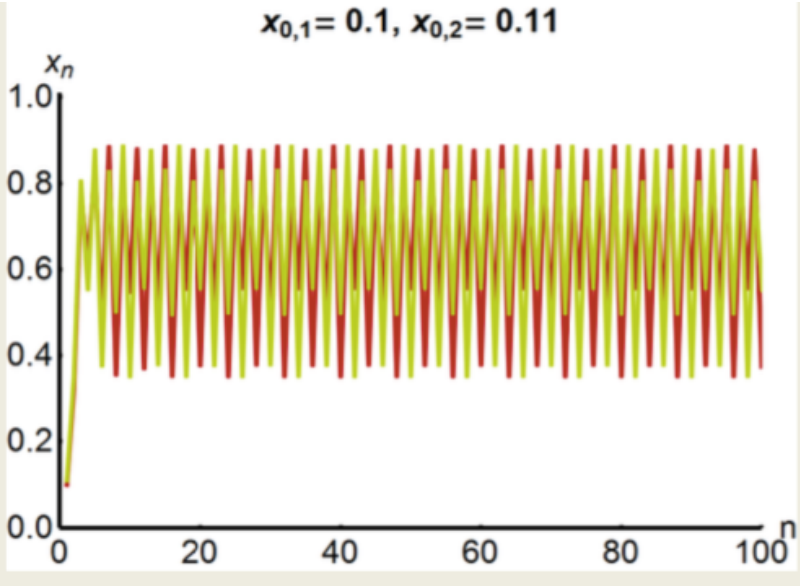
"Regular" behavior, periodic attractors

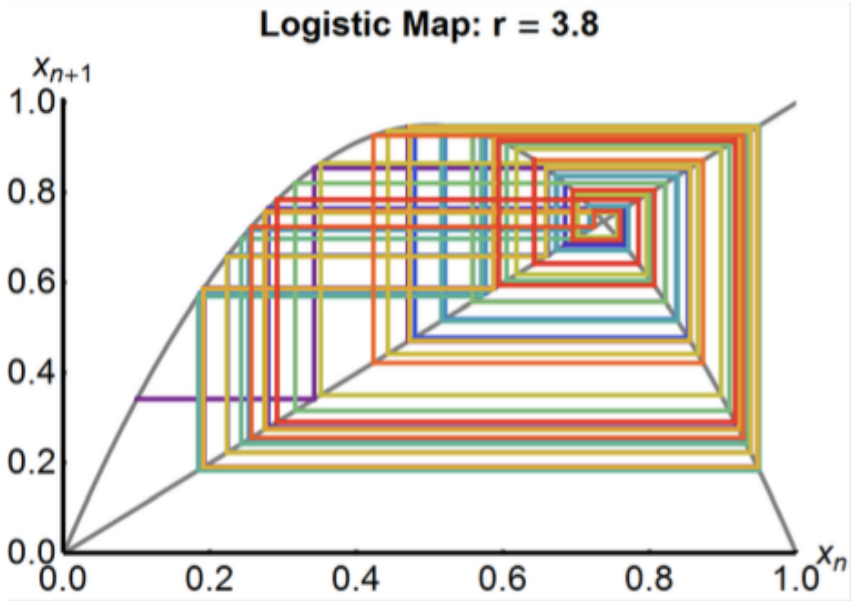


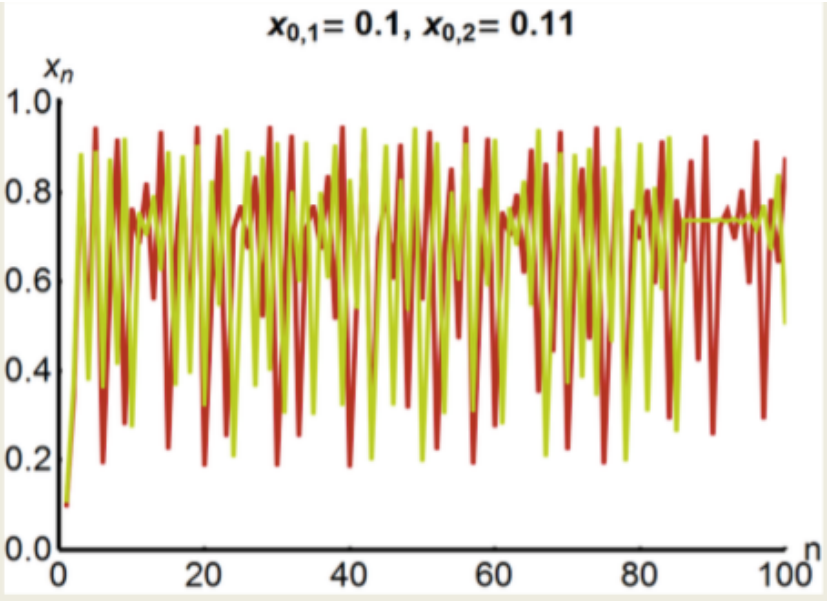


Transition to chaotic behavior

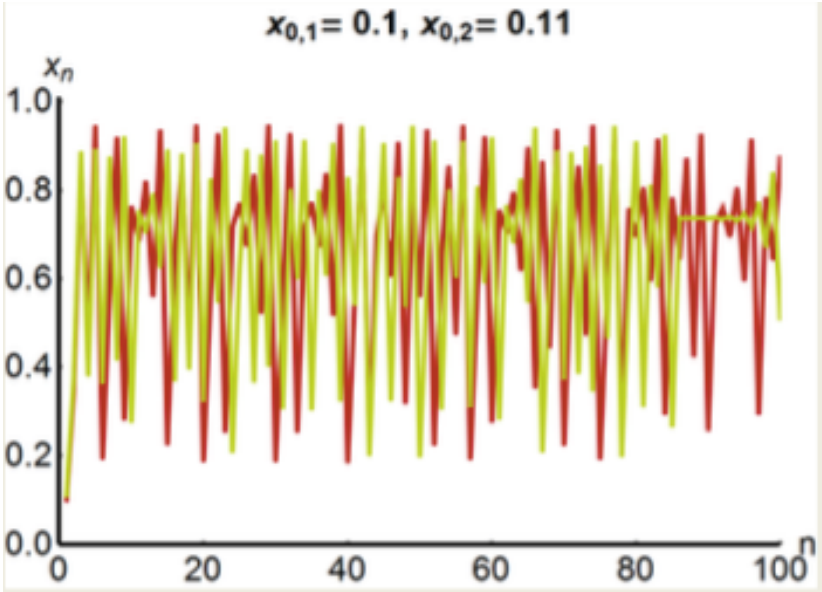


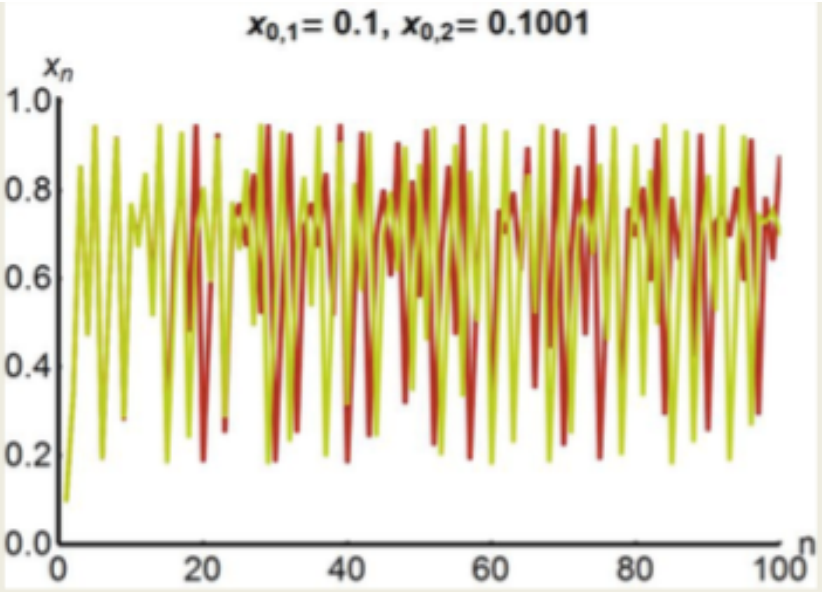


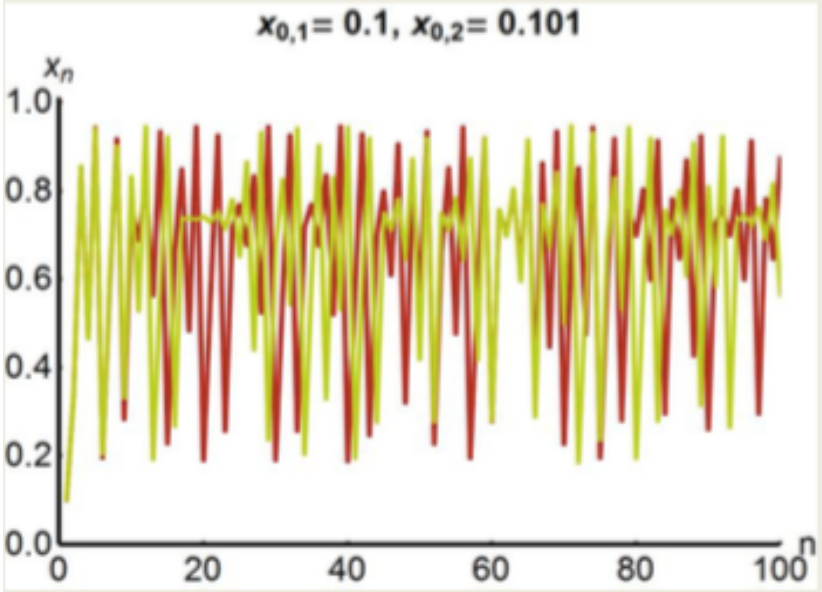


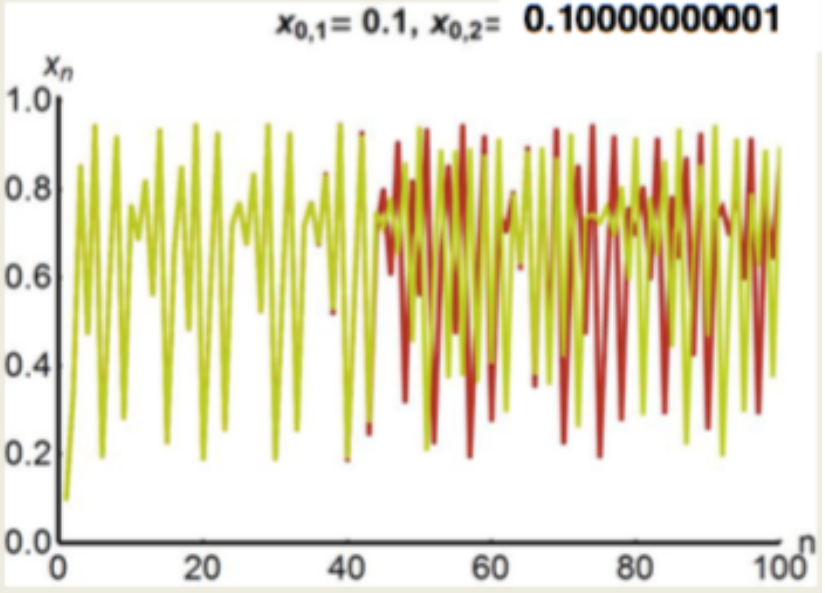


Chaos: Sensitivity to initial conditions



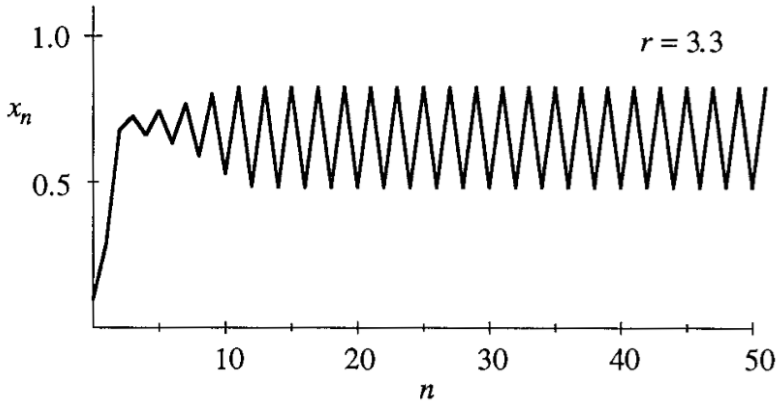




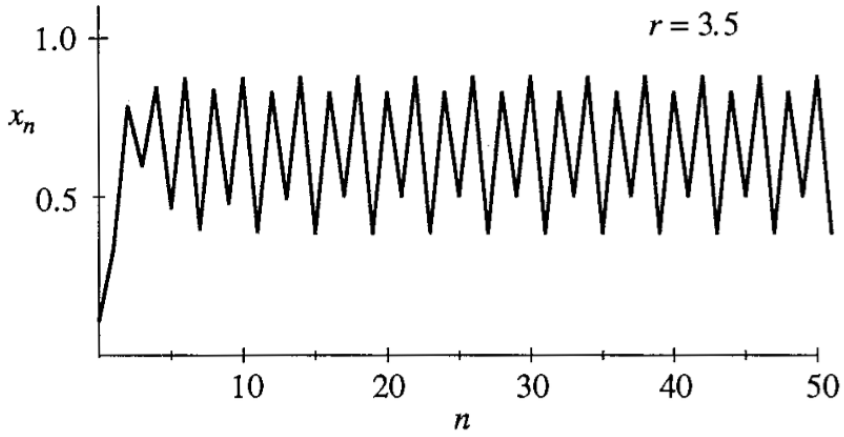


Periods in the logistic map

- *Oscillating* about the previous steady state, alternating between small and large populations
- **Period-2 cycle:** Oscillation repeats every two iterations, periodic orbit



- **Period-4 cycle**



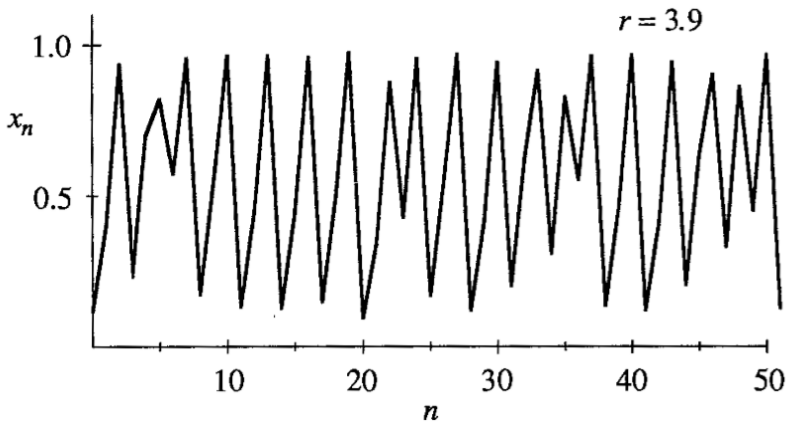
$r_1 = 3$	(period 2 is born)
$r_2 = 3.449...$	4
$r_3 = 3.54409...$	8
$r_4 = 3.5644...$	16
$r_5 = 3.568759...$	32
⋮	⋮
$r_\infty = 3.569946...$	∞

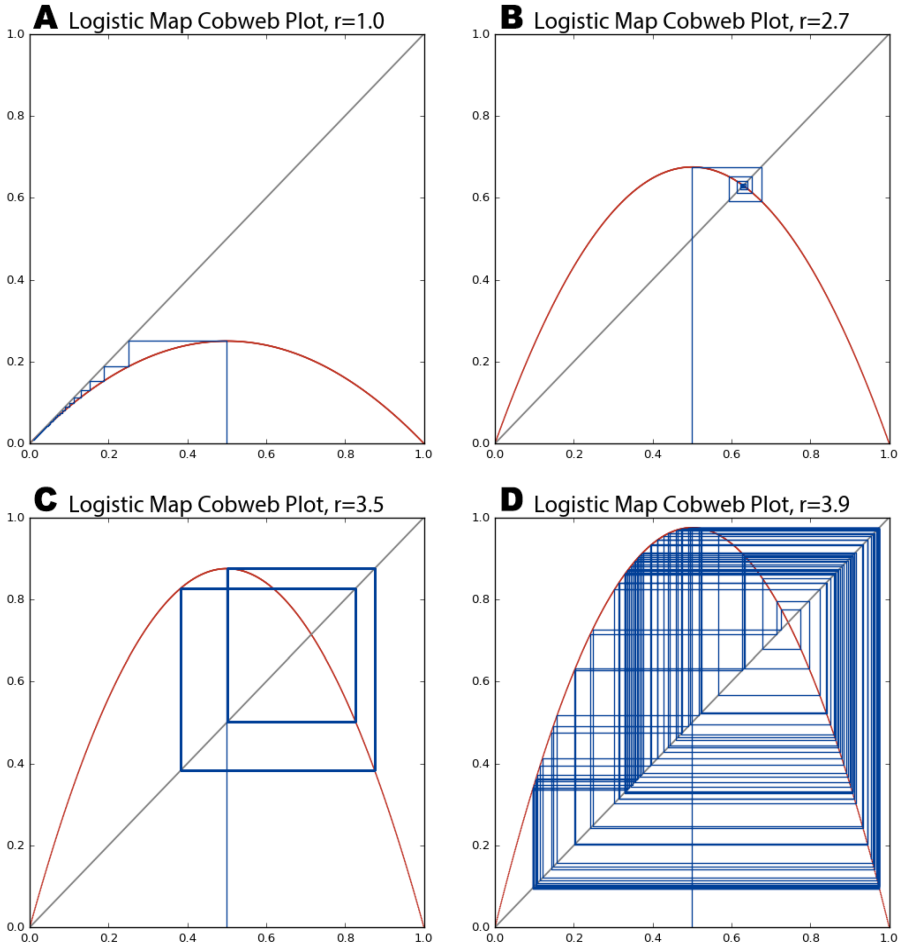
- **Period-doubling** to cycles appears by increasing r
- They correspond to **bifurcations i**n phase diagram
- Successive bifurcations come faster and faster!
- Limiting value $r_n \to r_\infty = 3.569946\ldots$
- Geometric convergence, in the limit the distance between successive values shrink to a constant:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots.$$

Chaos …

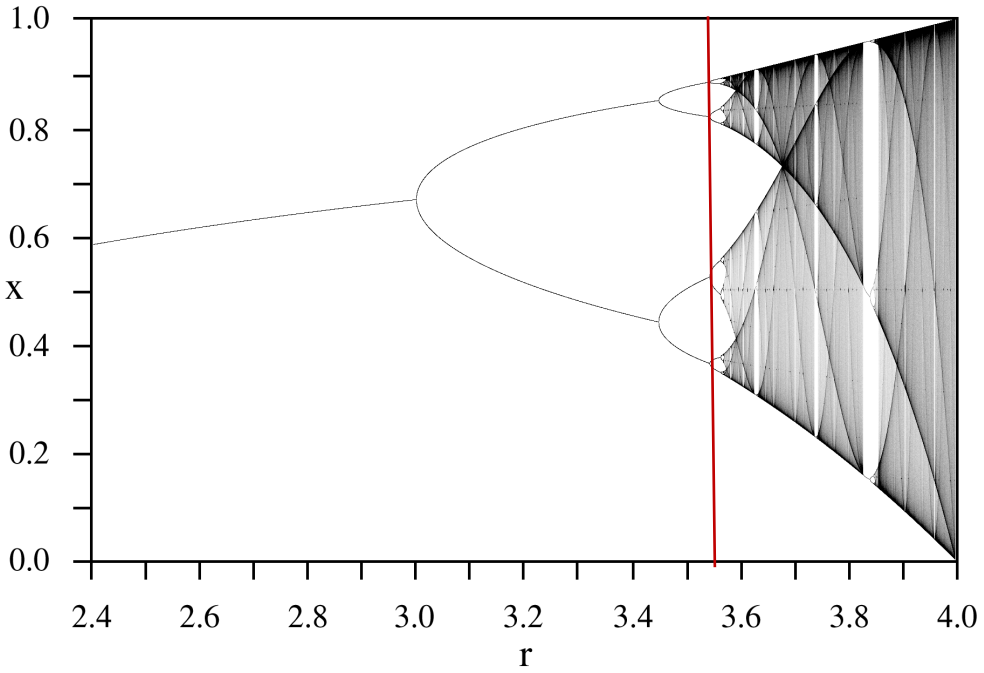
- $r > r_\infty$?
- For many values of r, the sequence never settles down to a fixed point or a periodic orbit
- **Aperiodic, bounded behavior!**





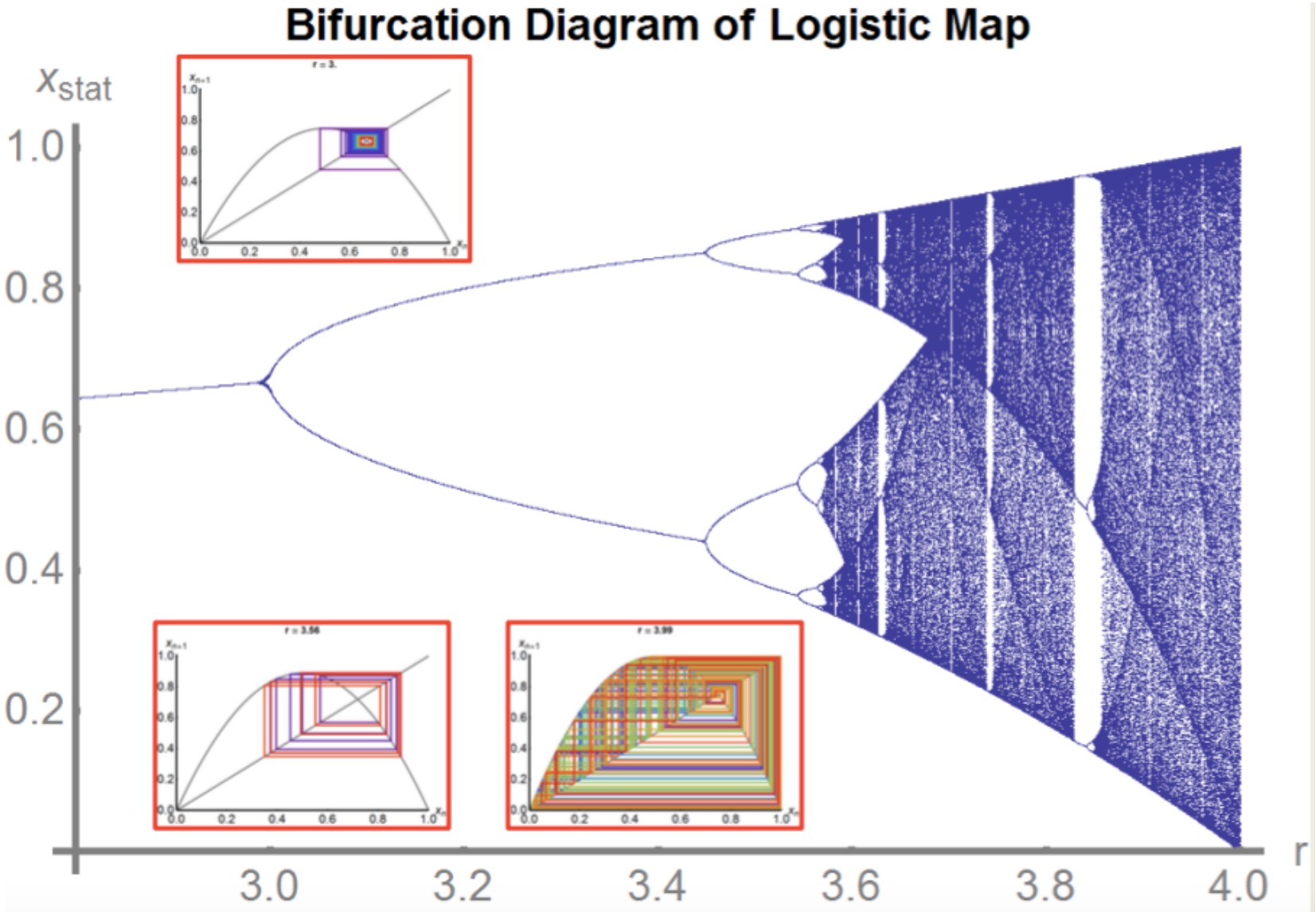
Orbit diagram

- What happens for larger r? Sure, more *chaos*.... Even more interesting things!
- **Orbit diagram:** system's attractors as a function of r
- *Construction:*
  - Choose a value of r
  - Select a random initial condition x_0 and generate the orbit: lets iterate for ~300 cycles to let the system settle down, then plot the next ~300 points from the map iterations
  - Move to an adjacent value of r and repeat, sweeping the r interval

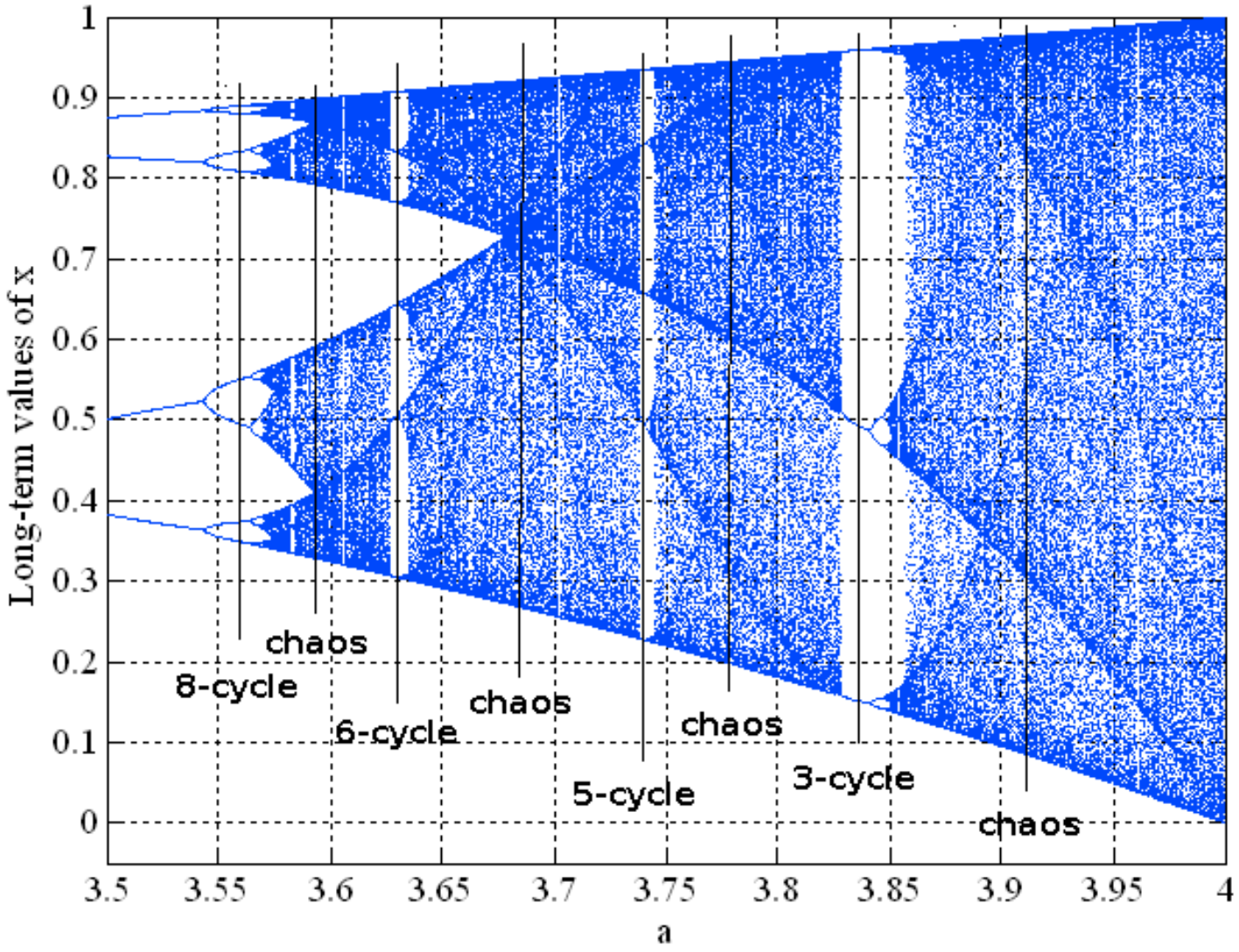


- At $r \approx r_\infty = 3.57$ the map becomes chaotic and the attractor changes from a finite to an infinite set of points
- For $r > 3.57$, mixture of order and chaos, with periodic windows interspersed between clouds of (chaotic) dots

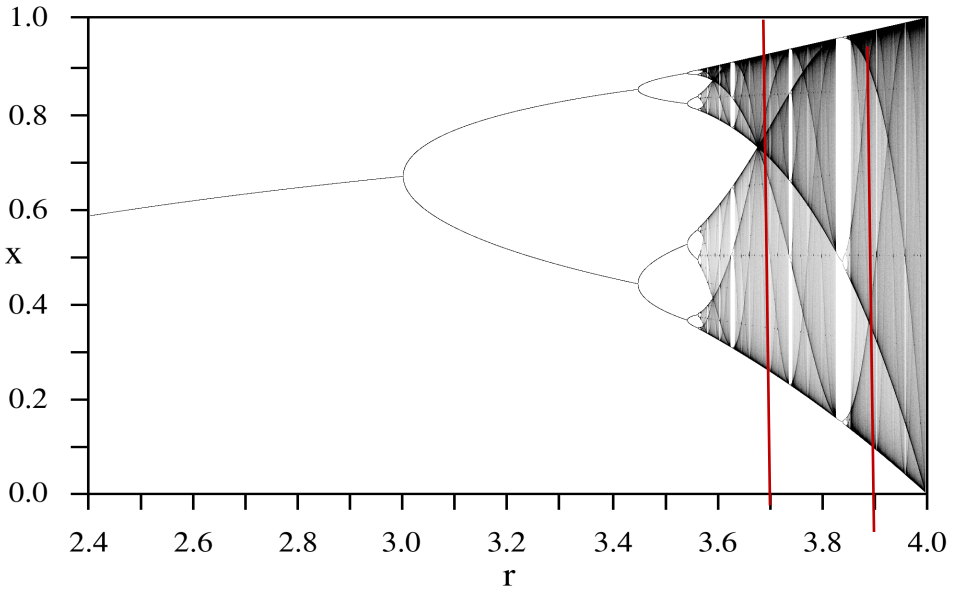
Orbit diagram



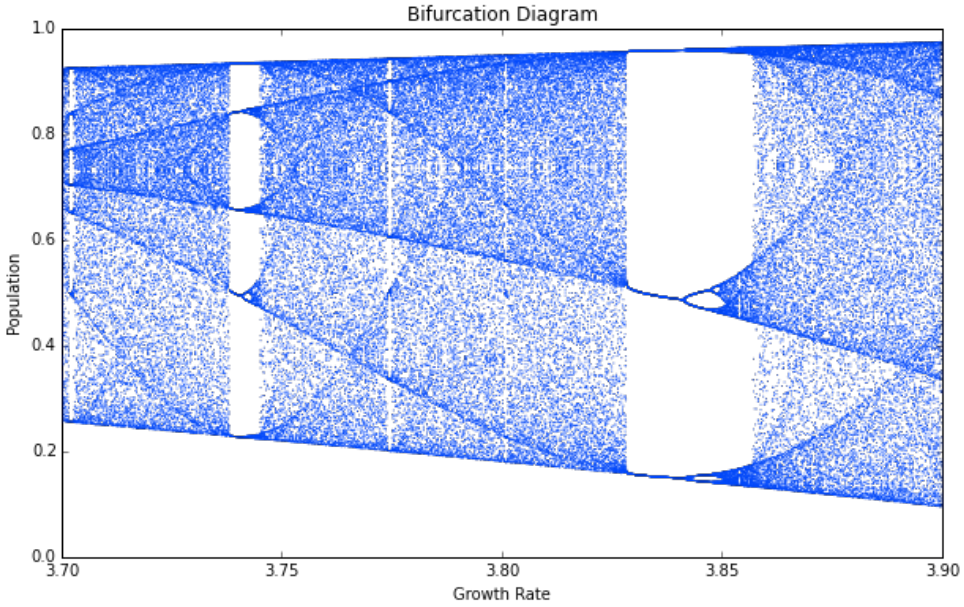
Chaos and order

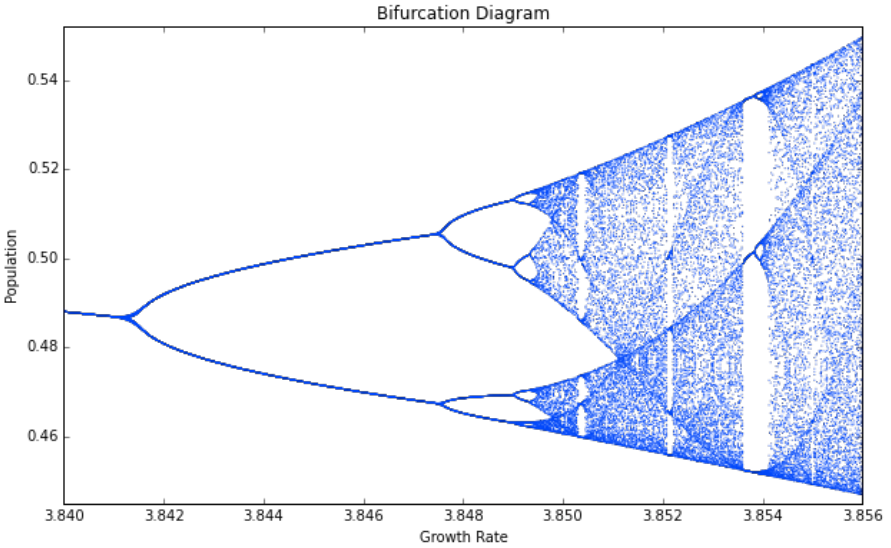


Chaos and order, Self-similarity



- The large window at $r \approx 3.83$ contains a stable period-3 orbit
- Looking at the period-3 window even closer: a copy of the orbit diagram reappears in miniature!
  → **Self-similarity**

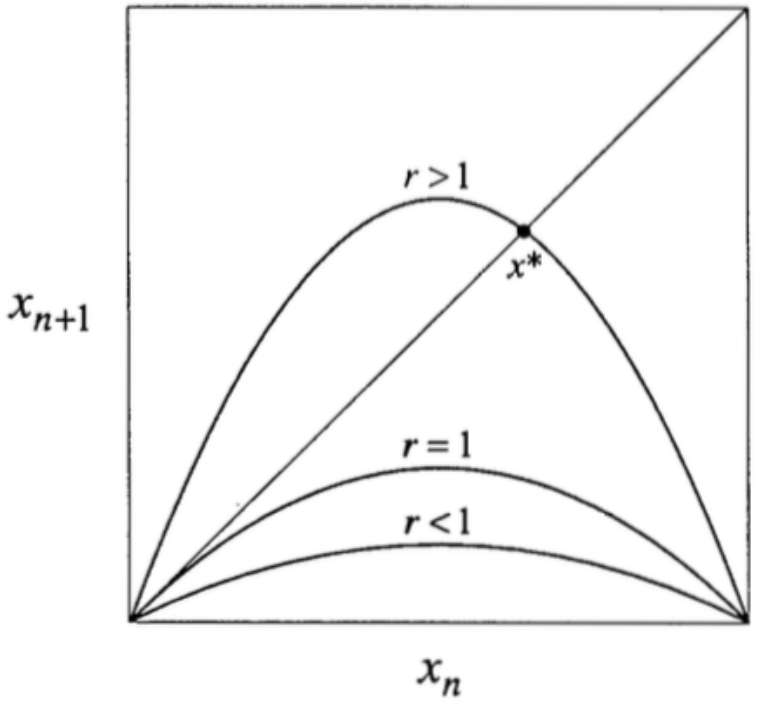




Logistic map, analysis

- $x_{n+1} = r x_n (1 - x_n)$, $0 \le r \le 4$, $0 \le x \le 1$, Fixed points and stability?
- Fixed points, are roots of: $x^* = f(x^*) = r x^* (1 - x^*)$ → $x^* = 0$, $x^* = 1 - \frac{1}{r}$
- Since $x \ge 0, x^*$ is in the range of allowable values only if $r \ge 1$
- Stability depends on multiplier $\lambda = f'(x^*) = r - 2 r x^*$
- $x^* = 0$: $f'(x^* = 0) = r$ → Origin is stable for $r < 1$, unstable for $r > 1$
- $x^* = 1 - \frac{1}{r}$: $f'\left(1 - \frac{1}{r}\right) = 2 - r$ → $1 - \frac{1}{r}$ is stable for $1 < r < 3$, unstable for $r > 3$

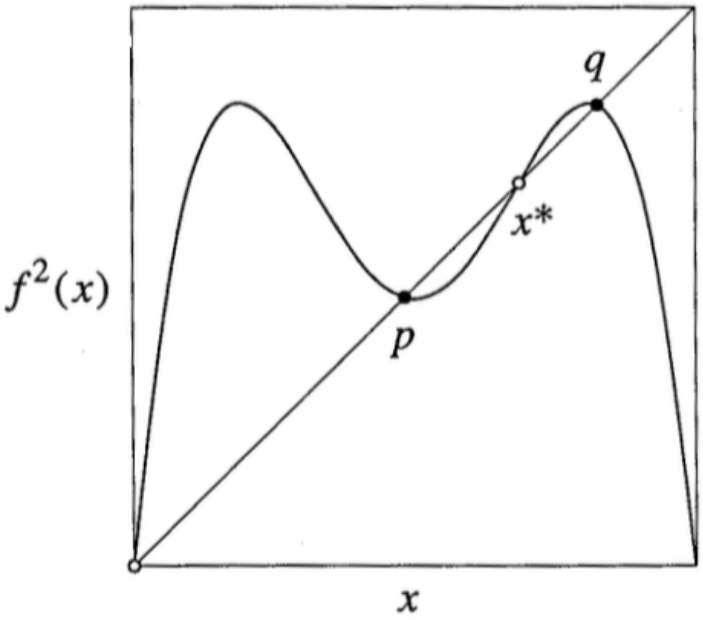
- For $r = 1$, a second fixed point appears, while the origin loses its stability
- -→ **Transcritical bifurcation** *at* $r = 1$
- When the slope of the parabola at $x^* = 0$ becomes too steep, the origin loses its stability (it happens at $r = 3$)
- → **Flip bifurcation** at $r = 3$, that are (usually) associated with period doubling and in this case a *2-period cycle is spawn*



Analysis: spawning of two-cycle

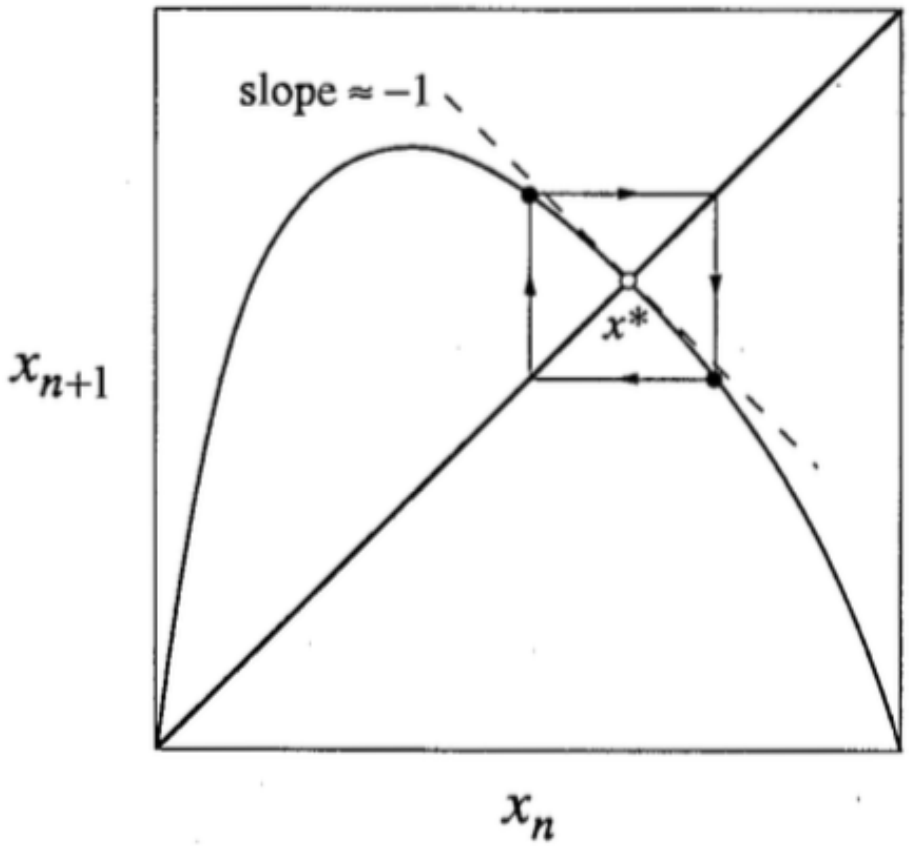
- The logistic map has a *two-cycle* for all $r > 3$

- Period-2 cycle: there are two states p and q, such that :
  - $f(p) = q,\ f(q) = p$, or equivalently,
  - $f\big(f(p)\big) = p$
  - → p (and q) fixed points of second-iterate map, $f^2(x) \equiv f\big(f(x)\big)$
- $f^2(x)$ is a quartic polynomial, that for $r > 3$ looks like:



- p, q corresponds to where the graph of $f^2(x)$ intersects the diagonal: $f^2(x) = x$
- … $p, q = \frac{r+1\sqrt{(r-3)(r+1)}}{2r}$, real for $r > 3$
- → A two-cycle exists for all $r > 3$
- At $r = 3$, the two-cycle *bifurcates* continuously from x^*

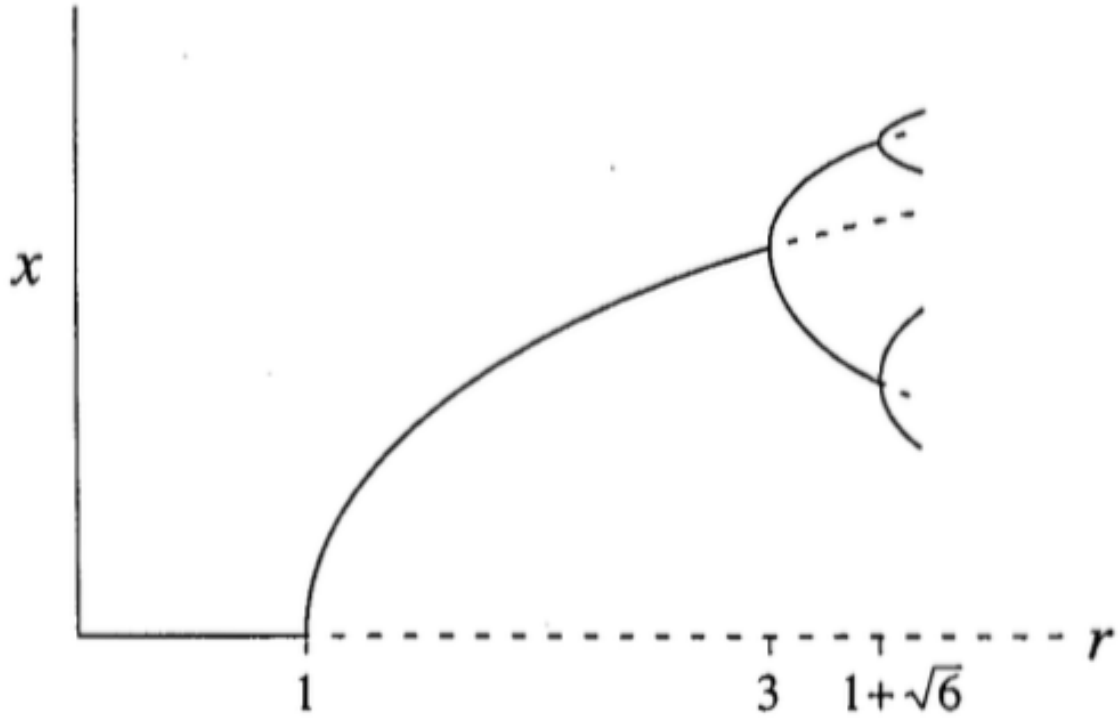
Flip bifurcations and period doubling



- If tangent slope, $f'(x^*) \approx -1$ and the graph of the function is concave near x^*, the cobweb tends to produce a small, stable 2-cycle around the fixed point
- The critical slope ≈ -1 corresponds to a **flip bifurcation** that gives rise two a 2-cycle
- How can we determine that the 2-cycle is stable or not?

- p, q are the solutions of $f^2(x) = x$ → The 2-cycle determined by p, q is stable iff p, q stable both stable fixed points of the f^2 map
- Doing the usual analysis ... for both p, q $\rightarrow \lambda = 4 - 2r - r^2$
- → The 2-cycle is stable iff $|4 - 2r - r^2| < 1$ → $r < 1 + \sqrt{6}$

A FIRST BIFURCATION DIAGRAM ...



- The dashed lines indicate fixed points that are instable
- The first bifurcation is a flip one, that creates a new equilibrium, losing the stability of the original one
- Each further pitchfork bifurcation is a supercritical one, with two new stable equilibrium points appearing and the original equilibrium losing its stability.