systems, such as the undamped pendulum, enabled us to draw the phase plane for this system and view it globally. In that case we were able to understand the motions for all initial conditions. Unfortunately, total energy is not a strict Lyapunov function for any equilibria of the damped system. Instead, we compute $\dot{E}(\mathbf{v}) = -by^2$, so that the inequality $\dot{E}(\mathbf{v}) < 0$ fails to be strict for points arbitrarily near the equilibrium.

Notice that trajectories (other than equilibrium points) do not stay on the x-axis, the set on which $\dot{E} = 0$. Therefore we might expect trajectories to behave as if E were a strict Lyapunov function. The following corollary, often called "LaSalle's Corollary" to the Lyapunov Theorem 7.23, not only provides another means of deducing asymptotic stability for equilibria of the damped system, but also gives information as to the extent of the basin of attraction for each asymptotically stable equilibrium. We postpone the proof of the theorem until Chapter 8 when we study limit sets of trajectories.

Corollary 7.26 (Barbashin-LaSalle) Let E be a Lyapunov function for $\bar{\mathbf{v}}$ on the neighborhood W, as in Definition 7.22. Let $Q = \{\mathbf{v} \in W : \dot{E}(\mathbf{v}) = 0\}$. Assume that W is forward invariant. If the only forward-invariant set contained completely in Q is $\bar{\mathbf{v}}$, then $\bar{\mathbf{v}}$ is asymptotically stable. Furthermore, W is contained in the basin of $\bar{\mathbf{v}}$; that is, for each $\mathbf{v}_0 \in W$, $\lim_{t\to\infty}(\mathbf{F}(t, \mathbf{v}_0)) = \bar{\mathbf{v}}$.

✎ **EXERCISE T7.17**

Let $\ddot{x} + b\dot{x} + \sin x = 0$, for a constant $b > 0$. (a) Convert the differential equation to a first-order system. (b) Use Corollary 7.26 to show that the equilibria $(n\pi, 0)$, for even integers n are asymptotically stable. (c) Sketch the phase plane for the associated first-order system.

7.7 LOTKA-VOLTERRA MODELS

A family of models called the Lotka-Volterra equations are often used to simulate interactions between two or more populations. Interactions are of two types. Competition refers to the possibility that an increase in one population is bad for the other populations: an example would be competition for food or habitat. On the other hand, sometimes an increase in one population is good for the other. Owls are happy when the mouse population increases. We will consider two cases of Lotka-Volterra equations, called competing species models and predator-prey models.

We begin with two competing species. Because of the finiteness of resources, the reproduction rate per individual is adversely affected by high levels of its own species and the other species with which it is in competition. Denoting the two populations by x and y, the reproduction rate per individual is

$$\frac{\dot{x}}{x} = a(1 - x) - by, \tag{7.46}$$

where the carrying capacity of population x is chosen to be 1 (say, by adjusting our units). A similar equation holds for the second population y, so that we have the **competing species** system of ordinary differential equations

$$\begin{aligned} \dot{x} &= ax(1 - x) - bxy \\ \dot{y} &= cy(1 - y) - dxy \end{aligned} \tag{7.47}$$

where $a, b, c,$ and d are positive constants. The first equation says the population of species x grows according to a logistic law in the absence of species y (i.e., when $y = 0$). In addition, the rate of growth of x is negatively proportional to xy, representing competition between members of x and members of y. The larger the population y, the smaller the growth rate of x. The second equation similarly describes the rate of growth for population y.

The **method of nullclines** is a technique for determining the global behavior of solutions of competing species models. This method provides an effective means of finding trapping regions for some differential equations. In a competition model, if a species population x is above a certain level, the fact of limited resources will cause x to decrease. The nullcline, a line or curve where $\dot{x} = 0$, marks the boundary between increase and decrease in **x**. The same characteristic is true of the second species y, and it has its own curve where $\dot{y} = 0$. The next two examples show that the relative orientation of the x and y nullclines determines which of the species survives.

EXAMPLE 7.27

(Species extinction) Set the parameters of (7.47) to be $a = 1, b = 2, c = 1,$ and $d = 3$. To construct a phase plane for (7.47), we will determine four regions in the first quadrant: sets for which (I) $\dot{x} > 0$ and $\dot{y} > 0$; (II) $\dot{x} > 0$ and $\dot{y} < 0$; (III) $\dot{x} < 0$ and $\dot{y} > 0$; and (IV) $\dot{x} < 0$ and $\dot{y} < 0$. Other assumptions on $a, b, c,$ and d lead to different outcomes, but can be analyzed similarly.

In Figure 7.18(a) we show the line along which $\dot{x} = 0$, dividing the plane into two regions: points where $\dot{x} > 0$ and points where $\dot{x} < 0$. Analogously, Figure 7.18(b) shows regions where $\dot{y} > 0$ and $\dot{y} < 0$, respectively. Combining the information from these two figures, we indicate regions (I)–(IV) (as described above) in Figure 7.19(a). Along the **nullclines** (lines on which either $\dot{x} = 0$ or

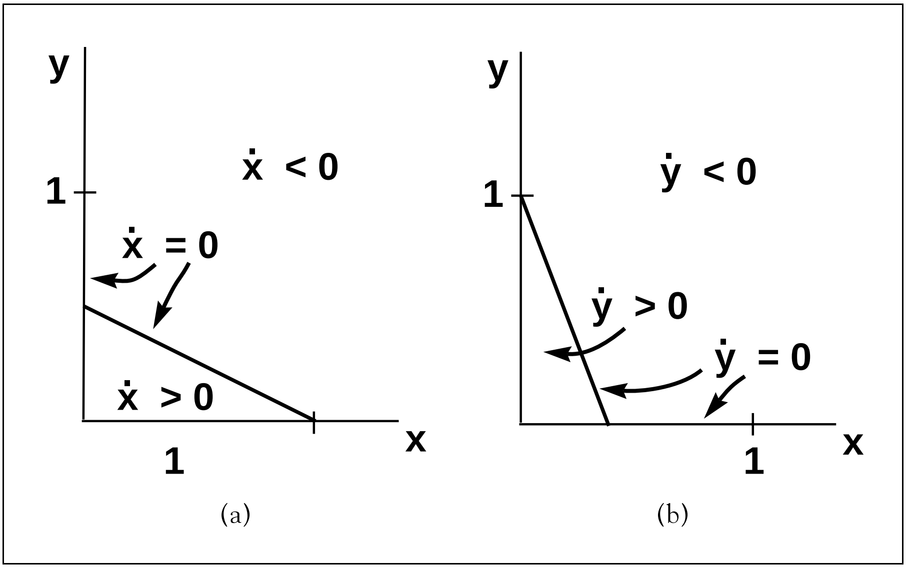

Figure 7.18 Method of nullclines for competing species.
The straight line in (a) shows where $\dot{x} = 0$, and in (b) it shows where $\dot{y} = 0$ for $a = 1$, $b = 2$, $c = 1$, and $d = 3$ in (7.47). The y-axis in (a) is also an x-nullcline, and the x-axis in (b) is a y-nullcline.

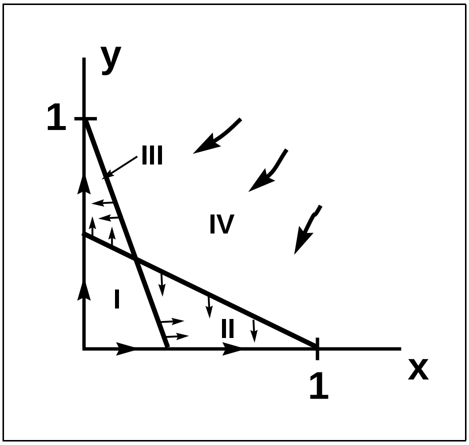

Figure 7.19 Competing species: Nullclines.
The vectors show the direction that trajectories move. The nullclines are the lines along which either $\dot{x} = 0$ or $\dot{y} = 0$. In this figure, the x-axis, the y-axis, and the two crossed lines are nullclines. Triangular regions II and III are trapping regions.

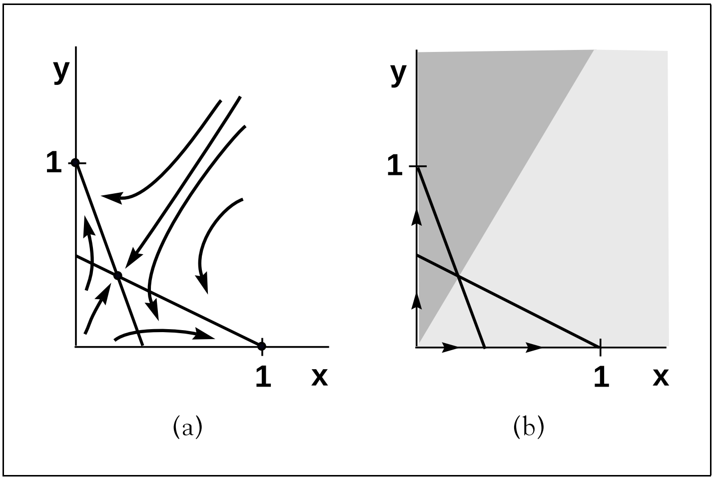

Figure 7.20 Competing species, extinction.
(a) The phase plane shows attracting equilibria at $(1, 0)$ and $(0, 1)$, and a third, unstable equilibrium at which the species coexist. (b) The basin of $(0, 1)$ is shaded, while the basin of $(1, 0)$ is the unshaded region. One or the other species will die out.

$\dot{y} = 0$), arrows indicate the direction of the flow: left/right where $\dot{y} = 0$ or up/down where $\dot{x} = 0$. Notice that points where the two different types of nullclines cross are equilibria. There are four of these points. The equilibrium $(0, 0)$ is a repellor; $(1/5, 2/5)$ is a saddle; and $(1, 0)$ and $(0, 1)$ are attractors.

The entire phase plane is sketched in Figure 7.20(a). Almost all orbits starting in regions (I) and (IV) move into regions (II) and (III). (The only exceptions are the orbits that come in tangent to the stable eigenspace of the saddle $(1/5, 2/5)$. These one-dimensional curves form the "stable manifold" of the saddle and are discussed more fully in Chapter 10.) Once orbits enter regions (II) and (III), they never leave. Within these trapping regions, orbits follow the direction indicated by the derivative toward one or the other asymptotically stable equilibrium. The basins of attraction of the two possibilities are shown in Figure 7.20(b). The stable manifold of the saddle forms the boundary between the basin shaded gray and the unshaded basin. We conclude that for almost every choice of initial populations, one or the other species eventually dies out.

EXAMPLE 7.28

(Coexistence) Set the parameters to be $a = 3$, $b = 2$, $c = 4$, and $d = 3$. The nullclines are shown in Figure 7.21(a). In this case there is a steady state at $(x, y) = (2/3, 1/2)$ which is attracting. The basin of this steady state includes

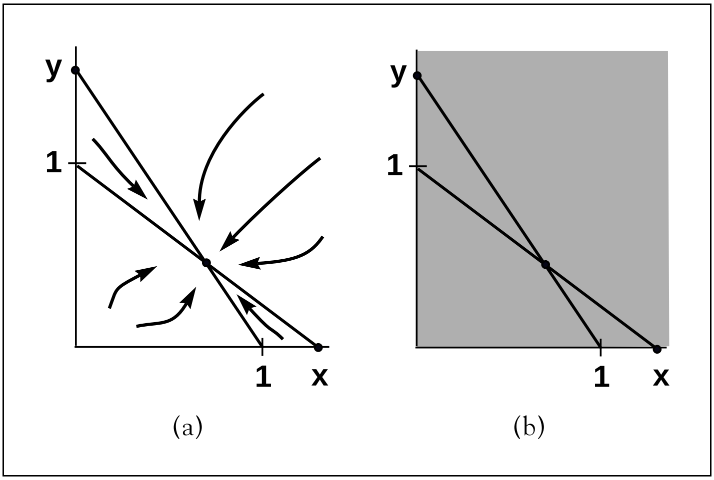

Figure 7.21 Competing species, coexistence.
(a) The phase plane shows an attracting equilibrium in which both species survive. The x-nullcline $y = \frac{3}{2} - \frac{3}{2}x$ has smaller x-intercept than the y-nullcline $y = 1 - \frac{3}{4}x$. According to Exercise T7.18, the equilibrium $(\frac{2}{3}, \frac{1}{2})$ is asymptotically stable. (b) All initial conditions with $x > 0$ and $y > 0$ are in the basin of this equilibrium.

the entire first quadrant, as shown in Figure 7.21(b). Every set of nonzero starting populations moves toward this equilibrium of coexisting populations.

✎ **EXERCISE T7.18**

Consider the general competing species equation (7.47) with positive parameters a, b, c, d. (a) Show that there is an equilibrium with both populations positive if and only if either (i) both a/b and c/d are greater than one, or (ii) both a/b and c/d are less than one. (b) Show that a positive equilibrium in (a) is asymptotically stable if and only if the x-intercept of the x-nullcline is less than the x-intercept of the y-nullcline.

EXAMPLE 7.29

(Predator-Prey) We examine a different interaction between species in this example in which one population is prey to the other. A simple model of this interaction is given by the following equations:

$$\begin{aligned} \dot{x} &= ax - bxy \\ \dot{y} &= -cy + dxy \end{aligned} \tag{7.48}$$

where a, b, c, and d are positive constants.

✎ EXERCISE T7.19

Explain the contribution of each term, positive or negative, to the predator-prey model (7.48).

System (7.48) has two equilibria, $(0, 0)$ and $(\frac{c}{d}, \frac{a}{b})$. There are also nullclines; namely, $\dot{x} = 0$ when $x = 0$ or when $y = \frac{a}{b}$, and $\dot{y} = 0$ when $y = 0$ or when $x = \frac{c}{d}$. Figure 7.22(a) shows these nullclines together with an indication of the flow directions in the phase plane.

Unlike previous examples, there are no trapping regions. Solutions appear to cycle about the nontrivial equilibrium $(\frac{c}{d}, \frac{a}{b})$. Do they spiral in, spiral out, or are they periodic? First, check the eigenvalues of the Jacobian at $(\frac{c}{d}, \frac{a}{b})$.

✎ EXERCISE T7.20

Find the Jacobian **Df** for (7.48). Verify that the eigenvalues of $\mathbf{Df}(\frac{c}{d}, \frac{a}{b})$ are pure imaginary.

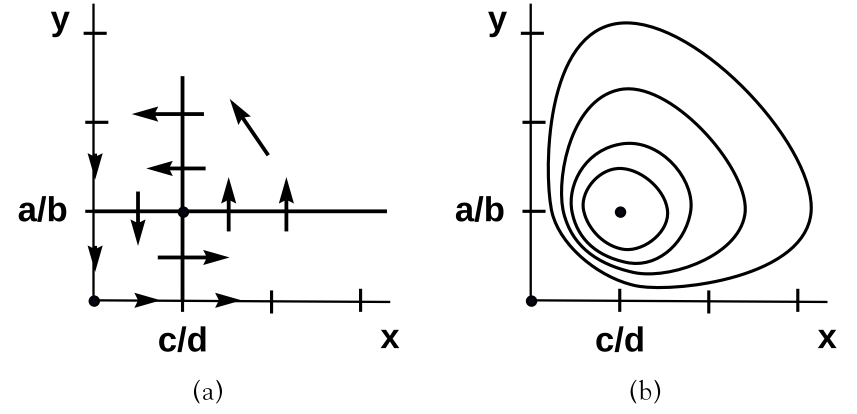

Figure 7.22 Predator-prey vector field and phase plane.
(a) The vector field shows equilibria at $(0, 0)$ and $(\frac{c}{d}, \frac{a}{b})$. Nullclines are the x-axis, the y-axis, and the lines $x = \frac{c}{d}$ and $y = \frac{a}{b}$. There are no trapping regions. (b) The curves shown are level sets of the Lyapunov function E. Since $\dot{E} = 0$, solutions starting on a level set must stay on that set. The solutions travel periodically around the level sets in the counterclockwise direction.

Since the system is nonlinear and the eigenvalues are pure imaginary, we can conclude nothing about the stability of $(\frac{c}{d}, \frac{a}{b})$. Fortunately, we have a Lyapunov function.

✎ EXERCISE T7.21

Let

$$E(x, y) = dx - c \ln x + by - a \ln y + K,$$

where $a, b, c,$ and d are the parameters in (7.48) and K is a constant. Verify that $\dot{E} = 0$ along solutions and that E is a Lyapunov function for the equilibrium $(\frac{c}{d}, \frac{a}{b})$.

We can conclude that $(\frac{c}{d}, \frac{a}{b})$ is stable. Solutions of (7.48) lie on level curves of E. Since $(\frac{c}{d}, \frac{a}{b})$ is a relative minimum, these level curves are closed curves encircling the equilibrium. See Figure 7.22(b). Therefore, solutions of this predator-prey system are periodic for initial conditions near $(\frac{c}{d}, \frac{a}{b})$. In fact, every initial condition with x and y both positive lies on a periodic orbit.