Homework 1
Stability analysis of non-linear dynamical systems

(Max score: 125)

15-382: Collective Intelligence (Spring 2019)

OUT: February 5, 2019

DUE: February 15, 2019 at 11:55pm - Available late days: 1

Instructions

The homework consists of a *main* section, which is the Section 1, and an *optional* one, which is Section 2. This second section has been included to let the student practicing, if desired, with some basic notions and concepts about ODEs. Neither the included ODE theory nor the excercises pretended to cover adequately the topic. However, some practice and exposure to the practice of ODE is useful in general and can serve as a starting point for further studies in the field (the textbook mentioned in the course website can be used as a reference in this respect).

Homework Policy

Homework is due on Autolab by the posted deadline. As a general rule, you have a total of 6 late days. For this homework you cannot use more than 1 late day. No credit will be given for homework submitted after the late day. After your 6 late days have been used you will receive 20% off for each additional day late.

If you find a solution in any source other than the material provided, you must mention the source.

Submission

Create a zipped archive including a PDF file with the answers to the provided questions (they can be handwritten, but in this case you must have / use a "readable" handwriting). The zipped archive should be submitted to Homework 1 on autolab.

Contents

1 Analysis of a dynamical model of competing species

In some closed environment there are *two different populations*, 1 and 2, that do not mate with each other (e.g., different animal species, closed human groups/societies, different capital assets, antigens-antibodies immune systems).

In the absence of mutual interaction, a reasonable model for describing the dynamical evolution of the system of two populations (in terms of how many individuals are there), is to individually apply a *logistic growth* model, that accounts for the capacity of the environment in terms of how many individuals of each type it can contain. In this case, the dynamical system that models populations' evolution over time is:

$$\begin{cases} \frac{dx_1}{dt} = x_1(g_1 - c_1 x_1) \\ \frac{dx_2}{dt} = x_2(g_2 - c_2 x_2) \end{cases} \tag{1}$$

where, g_1, g_2 are the (free capacity) growth rates, and c_1, c_2, are the capacity limits imposed by the environment (e.g., because of a limited food supply, or capital, or energy). Notice that the state variables $x_i, i = 1, 2$ expresses the *fraction* of the population i (at time t) with respect to the maximal population i that the environment can hold.

If the two populations *compete* for a common, vital resource, the logistic model is not sufficient to describe systems' evolution. However, it can still be used by adding a non-linear *correction* that accounts for the negative interference between the populations, obtaining the following new model:

$$\begin{cases} \frac{dx_1}{dt} = x_1(g_1 - c_1 x_1 - i_1 x_2) \\ \frac{dx_2}{dt} = x_2(g_2 - c_2 x_2 - i_2 x_1) \end{cases} \tag{2}$$

where the coefficient i_1 represents the negative impact that the presence of population 2 has on population 1, and, symmetrically, i_2 is the negative impact that the presence of population 2 has on population 1. More precisely, they define the rate of decrease of one population proportionally to the current magnitude of the other population. All coefficients are positive (that is necessary to make fully sense of the negative signs).

As defined, the model is clearly non-linear, that can pose challenges finding analytic solutions and studying the stability of the equilibrium points. However, in order to control the overall stability of the environment, it is of *strategic importance* to get a good understanding of what dynamics are to be expected, and how it's potentially possible to intervene on the system (e.g., to avoid that one animal species disappears from an ecosystem).

Software module for the analysis of dynamical systems

At this aim, a number of tools for the analytic and numeric analysis of linear and non linear systems of ODEs are provided and *must be used* for the homework. The tools are in the form of a Python class, `DynamicalSystem`, that is included (together with examples of use) in the provided file `ode-analysis.py`. Class methods allow to: solve an ODE system (both symbolically and numerically), compute eigenvalues and eigenvectors of linear systems, find critical points, compute Jacobians (both symbolically and numerically), plot vector and flow fields. The class method `nonlinear_model_competing_species()` implements the dynamical system of question 1.1 and it is the non-linear system referred to in the `main()` part of the code. A number of linear systems are implemented and can be called by the given reference name (see the method `linear_model()`) in the `main()` setion about linear models. You are free to play with the code to experiment with both linear and non-linear systems. The code is written in `python 3` and the recommended python environment is `Anaconda`, https://docs.anaconda.com/anaconda/navigator/, that should provide all the required packages (at the moment the program should be run as a script from the command line, in this way it will correctly display the images with flows and vector fields, as well as will save them on the local folder as .png images).

1.1 Analysis of a specific instance of the model (50 points)

Given the following instance of the general model 2 above:

$$\begin{cases} \frac{dx_1}{dt} = x_1(1 - x_1 - \frac{1}{3}x_2) \\ \frac{dx_2}{dt} = x_2(\frac{3}{4} - x_2 - \frac{1}{2}x_1) \end{cases}$$

Let's assume that the model describes two populations of two distinct animal species competing for food supply in a (mostly) closed environment (e.g., a lake).

Your task is to make a complete study of the system, that includes:

1. Find the critical points.

2. Study the characteristics of stability of the critical points using a linearization approach (that, in turn, requires to linearize the system, and study the eigenspaces of the resulting linear system). For each equilibrium point:

   (a) report the equations of the linearized system;

   (b) report eigenvalues and eigenvectors;

   (c) report the general solution of the linearized system about the point;

   (d) define its characteristics of stability (asymptotically stable / unstable);

   (e) identify the name of the point in the adopted classification (e.g., saddle, improper node);

   (f) discuss / describe the geometry of the flows about the equilibrium point (e.g., compute the slopes of the flows in order to explain where they move and how fast / slow), and relates it to the evolution of the two physical populations;

   (g) discuss the ecological meaning of the equilibrium (e.g., something like "equilibrium point (0,1) means that population 1 gets extinct, and according to the geometry of the trajectories, this solution happens for all trajectories starting within region A of the phase space, the velocity of convergence is however slow because of the elliptic form of the curves, ...").

3. Plot both the vector fields and the flows of the system (plots must be well readable and include all the relevant information).

4. Study the nullclines of the system:

   (a) define the equations of the nullclines and plot them in the vector field;

   (b) report the point where do they intersect;

   (c) discuss the general realation that the nullclines have with the flows;

   (d) because of their properties, each nullcline in practice divides the phase space in two regions (this can be well observed by looking at the vector field plot with the nullclines): explain and discuss this fact in ecological terms.

1.2 Analysis of the parametric form of the model (50 points)

Let's consider the system in its general form, as provided in Eq. 2. The goal is to define its stability properties depending on the choice of the parameters.

1. Find the critical points as a function of the parameters. These are in correspondence of the roots of the vector field equations. Finding the solutions of this system of two non-linear equations by a direct approach is indeed relatively complex and doesn't bring any insight. We can proceed in a simpler way by making a few observations that help to simplify the way to the solution. The system has four solutions (the equilibrium points are the solutions of a system of two second order equations). Looking at the equations, it would be clear that one solution corresponds to the origin, while another solution corresponds to the solutions of the two nullcline equations, to where they intersect, factoring out the cases for $x_1 = 0$ and $x_2 = 0$. The other two solutions corresponds to where one population gets extinct, that is, by imposing first $x_1 = 0$ and then $x_2 = 0$. Following these hints, it should be fairly easy to find, one by one, the four critical points

2. Study the characteristics of stability of the critical points (using a linearization approach) vs. the parameters.

   (a) write down the general expression of the Jacobian.

(b) since from an ecological point of view the only interesting equilibrium is that when both populations coexist, only consider this point for the analysis. (note: if you will write the Jacobian in a correct way, you will be able to notice that some of its entries contain the equations of the nullclines; since the critical point that we are considering should precisely be the one that corresponds to where the nullclines intersect, that is, are both zero, those parts of the Jacobian can be removed, simplifying a lot the form of the Jacobian matrix and of the consequent analysis).

    Use the Jacobian to find the eigenvalues of the linearized system and perform the eigenanalysis, describing the solutions (i.e., the stability properties) that result as a function of the parameters.

(c) relate and discuss the results from an ecological point of view (e.g., "when the mutual interference between the two populations is lower that the growth rates, than a stable solution can arise, otherwise one of the two populations get extinct"). Remember that the g parameters are a measure of intrinsic growth that a population can achieve alone. In a sense g is the inhibitory effect on growth that a population has on itself (due to limited resources), while the i parameter is the inhibitory effect due to the presence of the other population.

2 General questions on ODEs

The general form of an ODE of order n is the following:

$$f\Big(t, x(t), x'(t), x''(t), \ldots, x^{(n)}(t)\Big) = 0$$

where $x : I \subset \mathbb{R} \to \mathbb{R}$ is a real scalar function differentiable n times on the interval I, and f is a function of the $n+2$ functions in its argument.

If the function f is such that the ODE takes the general form:

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + a_{n-2}(t)x^{(n-2)} + \ldots + a_1 x' + a_0(t)x = g(t)$$

then we say that the ODE is *linear*, *non-linear* otherwise.

When the term $g(t)$ (the term that doesn't include any function of x or of its derivatives) is zero, than the equation is *homogeneous*, *non-homogeneous* otherwise.

If $g(t)$ and all the coefficients $a(t)$ are constants (i.e., no dependence on the independent variable t) then the equation is *autonomous*

For all the questions below, your answer must include the reasoning and/or the steps to derive the solution. Without these the answer won't be considered as valid.

2.1 Classification (5 points)

For each one of the following differential equations classify it according to order, linear / non-linear, homogeneous / non-homogeneous, autonomous or not:

1. $x^{(3)} + t^2 x' + x = 0$
2. $x'' + x^2 - t^3 - 1 = 0$
3. $x'' + tx = (t-1)\sin(t)$
4. $x^{(3)} + t^2 = 0$
5. $x^{(5)} + tx^2 = 0$
6. $x' + ty = t$

2.2 Transformation to a system of linear equations (5 points)

Any ODE of order n can be transformed into a system on n ODE of the first order. The transformation is operated by defining new variables $x_i, i = 1, \ldots, n-1$, where:

$$x_1 = x$$
$$x_2 = x'$$
$$x_3 = x''$$
$$\ldots$$
$$x_n = x^{(n-1)}$$

if we differentiate both sides, we get that $x_1' = x' = x_2$, $x_2' = x'' = x_3$, so on. Therefore we can operate the substitutions and get a system of equations. For instance, given $2x'' - 5x' + x = 0$, we can make the transformation:

$$x_1(t) = x(t)$$
$$x_2(t) = x'(t)$$

and by differentiating (x_1' becomes equal to x', and x_2' becomes equal to x'') and substituting into the original equation, we get the following system of two first order equations:

$$x_1'(t) = x_2$$
$$x_2'(t) = -\frac{1}{2}x + \frac{5}{2}x'$$

Transform the following equations into systems of equations of the first order:

1. $x^{(4)} + 3x'' - \sin(t)\, x' + 8x = t^2$
2. Use the same equation of the previous question and include also the initial conditions (i.e., you need to transform also the initial conditions): $x(0) = 1 \;\; x'(0) = 2 \;\; x''(0) = 3 \;\; x'''(0) = 4$

2.3 Solution concepts (5 points)

In a very general sense, the solution of an ODE is a function that satisfies the equation. For instance, given the equation $x' = 2x$, the function $y(t) = e^{2t}$ is a solution, as it can be verified by subustituting it into the equation: $y'(t) = 2e^{2t}$, therefore we obtain that the equality $y' = 2y$ is satisfied.

For a first order equation, a *Cauchy problem*, is a pair of equations: one equation defining an ODE and another equation defining the *initial condition* for the ODE.

In practice, the solution of a Cauchy problem consists in finding, among the general solutions of the ODE, the one that is compatible (passes by) the specified initial condition. In more general terms, an *initial value problem*, specifies the general order n ODE and the initial conditions for all the n derivatives.

For the following equations, verify that the given equation is a solution, or find the solution among the proposed ones:

1. Given $x'' + tx = (t-1)\sin(t)$, verify that $y(t) = \sin(t)$ is a solution
2. Given $x'' + t = 0$, which one of the following equations is a solution? $y(t) = t^2 + 1$, $y(t) = -t^3/6$, $y(t) = -t^3/6 + 2t$, $y(t) = t^3/6$
3. Given the Cauchy problem $\begin{cases} x' = x^2 \\ x(0) = 1 \end{cases}$ find the solution by first applying the method of separation of variables to find the general solution, and then using the initial condition to set the value of the integration constant and find in this way the unique solution.

   Is the found solution applicable over the entire interval of the real numbers for t? In other words, say over which interval of the independent variable t the solution can be used. (In general, the definition of the interval of validity of a solution function is a relatively complex issue in the solution of the ODEs).

2.4 Solution procedures (10 points)

Depending on the type of the ODE, different solution approaches are available. Below we consider only two special cases: separable equations and 1st order linear equations.

We have seen the method of *separation of the variables*, which is in general applicable to first order linear or non-linear equations of the form:

$$x'(t) = a(t) \cdot F(x(t))$$

where both a and F are continous functions. The solutions is obtained by the solution of the equation between two integrals:

$$\int \frac{1}{F(x)} dx = \int a(t) dt$$

In the case of a *linear equation of the first order*, the general form of the equation is:

$$x'(t) + a_0(t)x(t) = g(t)$$

where a_0 and g are real and continous functions. The general solution to such class of ODEs is:

$$x(t) = e^{-A(t)} \left[c_1 + \int g(t) e^{A(t)} dt \right]$$

where

$$A(t) := \int a_0(t) dt$$

In the case of a Cauchy problem, the initial condition is used to set the value of the integration constant(s).

Solve by using the above methods the following ODEs and Cauchy problems:

1. $x' = x^2 t^2$
2. $\begin{cases} x' = x^2 t^2 \\ x(1) = 3 \end{cases}$
3. $x'(t) = 2x(t) + 1$
4. $x'(t) + \cos(t)x(t) = \sin(2t)$