Homework 2

Modeling complex systems, Stability analysis, Discrete-time dynamical systems, Deterministic chaos

(Max useful score: 100 - Available points: 130)

15-382: Collective Intelligence (Spring 2019)

OUT: February 24, 2018

DUE: March 11, 2018 at 11:55pm - Available late days: 1

Instructions

Homework Policy

Homework is due on Autolab by the posted deadline. As a general rule, you have a total of 6 late days. For this homework you cannot use more than 1 late day. No credit will be given for homework submitted after the late day. After your 6 late days have been used you will receive 20% off for each additional day late.

If you find a solution in any source other than the material provided, you must mention the source.

Submission

Create a zipped archive including: a PDF file with the answers to the provided questions (they can be handwritten, but in this case you must have / use a "readable" handwriting), files that have been used for dealing with the questions that require programming, a README file that specifies how to use / run the programming files. The zipped archive should be submitted to Homework 2 on Autolab.

Contents

1 Stability study

1.1 Lyapounov functions, limit cycles (17 points)

Let's consider the van der Pol oscillator:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - \mu(x_1^2 - 1)x_2 \end{cases} \tag{1}$$

1. Determine whether the equilibrium point $(0, 0)$ is hyperbolic or not as function of the values of the parameter μ.

2. Use a Lyapounov function to study the stability of $(0, 0)$ as a function of the parameter μ (remember that you need to make a guess for the Lyapounov function).

3. Show in a plot the phase portraits for $\mu = -0.5$ and $\mu = 0.5$ (make use of the python software previously provided, considering plotting single trajectories).

4. Discuss the results of the plots: Do limit cycles appear? Are do they stable?

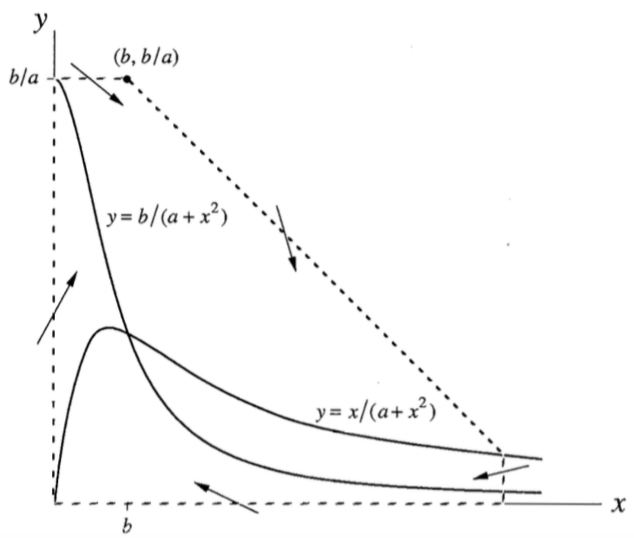
1.2 Existence of closed orbits, Poincaré-Bendixson theorem (18 points)

Glycolysis if a fundamental process that living cells adopts to obtain energy by breaking down sugar molecules. Under certain conditions, in intact yeast cells or muscle extracts glycolysis proceeds in an oscillatory fashion, with the concentrations of various intermediate products increasing and decreasing with a period of several minutes. A simple model to describe such oscillations in glycolysis is the following, where x and y are the concentrations of ADP and F6P, and $a, b > 0$ are kinetic parameters:

$$\begin{cases} \dot{x} = -x + ay + x^2 y \\ \dot{y} = b - ay - x^2 y \end{cases} \tag{2}$$

Since the model is meant to describe an oscillatory process that ends up into a stable reaction, we expect to find a *stable limit cycle* "somewhere" in the parameter space.

1. Find a *trapping region* in the $x - y$ plane. The trapping region is a candidate region where to apply the Poincaré-Bendixson theorem, to where the limit cycle would present. The region is actually given and shown in the figure. The two curves are the nullclines, while the dashed diagonal is a line of slope -1 that extends from point $(b, b/a)$ to the intersection with the nullcline $y = x/(a + x^2)$. You have to justify why the shown dashed region is actually a trapping region for the dynamical system.



2. The trapping region in the figure contains a fixed point. Where is it?

3. Determine for which values of a and b, the Poincaré-Bendixson theorem applies to the region (or modifications of it) and a closed orbit exists. Note that the PB theorem cannot be applied to the region as given because of the presence of an equilibrium point (which one?), but we can make the region being *not simply connected* and exclude that point (how?). However, even after doing this we need to check the conditions of the theorem, and one essential aspect consists in checking whether that equilibrium is an attracting or a repelling equilibrium. If it is a repelling one (e.g., an unstable node or a spiral), than nearby trajectories will end up in the trapping region (and stay there), while if it is an attracting one, trajectories from the region will escape the region and end up in the equilibrium, that would contradict the theorem. You need the make the above analysis more explicitly and asses formally whether the equilibrium is an attractive or a repelling one (hint: make use of the properties between trace and determinant of the linearized matrix in order to quickly and effectively check what is the nature of the equilibrium for different values of the parameters).

4. After answering the previous questions, you should have formally assessed that a limit cycle does exist within the trapping region. Can you sketch where it is and how the trajectories spiral towards it? (hint: use the previous information about the directions of the flows and the intuition about where the cycle could be in the non simply connected domain; you can also use the provided software to plot the vector field, that should precisely provide an idea of what's going on).

5. For any choice of a and b, would it be possible to observe a chaotic behavior in this two-dimensional system? Justify your answer.

2 Bifurcations

2.1 Characterization of bifurcation properties (12 points)

Which type of bifurcation best describes each one of the sentences in the list below?

1. A fixed point exists for all values of a parameter r but it changes its stability in correspondence of a critical value of r.

2. Two fixed points exist for all values of a parameter r but exchange their stability at the bifurcation point.

3. As r varies, two fixed points with different stability properties first coalesce into a half-stable fixed point and after disappear.

4. At a critical value of r, two new fixed points appear that inherits the stability of an existing fixed point; the latter keeps being a fixed point but loses its stability.

5. Fixed points tend to appear and disappear in pairs.

6. Increasing r after the critical point, blows up all the existing three fixed points into a single, unstable one.

7. This type of bifurcation does not exist in one dimensional systems.

2.2 Bifurcation study (13 points)

In both statistical mechanics and in neural networks, $\tanh(x)$ is a widely used function since it represents a bi-stable system, similarly to the logistic function. Therefore, the following dynamical system finds a number of applications:

$$\dot{x} = -x + \beta \tanh x$$

.

1. Show that the system undergoes a supercritical pitchfork bifurcation as β is varied. Note that the vector field is composed by the sum of two simple functions; exploit this fact to show *graphically* the situation as β varies (i.e., plot how the two functions overlap or not under different value for the parameter), and discuss on the basis of the graphs what happens in terms of the bifurcations.

2. Numerically, find the point x^* where the bifurcation happens (a fixed point is such that $x^* = f(x^*)$) and and show the bifurcation diagram x^* vs. β (hint: think of the fixed point x^* as the independent variable and compute / plot $\beta(x^*)$ in order to plot the (x^*, β) points).

3 Complete study of models expressed as iterated maps

3.1 One-population dynamics (20 points)

If a species breeds only at a particular time of the year, whether adults do or do not survive to breed in the next season has an important effect on population dynamics. Due to the existence of breeding seasons, the population growth, N_t, may be described by the following model:

$$N_{t+1} = N_t e^{r\left(1-\frac{N_t}{C}\right)}$$

where r and C are population-dependent constants expressing respectively, the intrinsic growth rate of the population and the carrying capacity of the environment.

1. Describe in words the mathematical model as you have to explain it to a politician: what the equations model (in terms of population's dynamics), what the the limits and the assumptions behind such a simple model.

2. Transform the given model into a dimensionless one. Discuss why (in general when dealing with population models) we want to do such a transformation.

   Note: All the questions below have to be answered using the dimensionless model.

3. Identify the equilibrium points of the model.

4. Consider the case for $r = 1.8$ and "dimensionless" initial population values $n_0 = 0.1$ and $n_0 = 1.5$: draw the cobweb plot[1] and perform an analysis of the stability of the equilibrium points.

5. Make a general analysis of the stability of the equilibrium points as a function of r (the analysis should agree with the conclusions that you have found using the cobweb plot for the specific case of $r = 1.8$).

6. Now, let's consider a different model for one-population dynamics, expressed as follows, where a is a positive parameter:

$$N_{t+1} = \frac{rN_t}{\left(1 + \frac{N_t}{C}\right)^a} \ .$$

   "Explain" the model, the role of a, and point out its differences and similarities with respect to the previous model.

3.2 Two-populations dynamics: host-parasitoid model (15 points)

A parasitoid is an insect having a lifestyle intermediate between a parasite and a usual predator. Parasitoid larvae live inside their hosts, feeding on the host tissues and generally consuming them almost completely. If we assume that the host has discrete nonoverlapping generations and can be attacked by a parasitoid during some interval of its life cycle, a simple model that could describe the host-parasitoid dynamics is the following,

[1]This will require that you write a little piece of python code that does the job, that should require a short coding time. In the unlikely case you aren't able to do it, I will provide a piece of code to you. In this case I will take out 3 point marks.

with $H_t \geq 0$ and $P_t \geq 0$ being, respectively, the host and parasitoid populations at time step t, and $r, c > 0$ constants:

$$\begin{cases} H_{t+1} = rH_t e^{-aP_t} \\ P_{t+1} = cH_t(1 - e^{-aP_t}) \end{cases} \tag{3}$$

1. As before, explain the model in words, and relates it to the previous, one-population, model.

2. At a time step t, what is the fraction of hosts that is not infested and can therefore live and reproduce? (The answer only requires looking at the model, no computations)

3. For a parasitoid, what is chance that a host is discovered (and infested)? (The answer only requires looking at the model, no computations)

4. If at time step t a new host is being "discovered" (infested) by parasitoids, how many new parasitoids are going to be found at time step $t + 1$? (The answer only requires looking at the model, no computations)

5. The model has two equilibrium points, in $(0, 0)$ and (H^*, P^*), where:

$$H^* = \frac{r \log r}{ac(r-1)}, \quad P^* = \frac{\log r}{a}.$$

   The eigenvalues of the Jacobian matrix at $(0, 0)$ are 0, and r. While, the eigenvalues at (H^*, P^*) are:

$$\frac{1}{2(r-1)}\left(r - 1 + \log r \pm i\sqrt{4(r-1)r\log r - (r - 1 + \log r)^2}\right).$$

   Describe the stability of the two equilibrium points and their ecological meaning.

6. Make a numerical study of the behavior of the system model for $r = 1, a = 0.001, c = 1$ (i.e., find and plot numerical solutions using the python class methods introduced in the previous homework). Consider as initial conditions values in the range $[800, 1400]$ for the hosts, and $[100, 400]$ for the parasitoids. Report the plot of the found solutions and discuss what happens. Would it be reasonable in a *real* ecosystem?

7. If you went through the previous question, you should have found that some unrealistic growing behavior happens. The following model, "corrects" the unrealistic growth of the previous one:

$$\begin{cases} H_{t+1} = H_t e^{r\left(1 - \frac{N_t}{C}\right) - aP_t} \\ P_{t+1} = cH_t\left(1 - e^{-aP_t}\right) \end{cases} \tag{4}$$

   Discuss why this new model should be more realistic. What kind of equilibrium do you expect this time? (Provide an answer based on the model, no computations)

4 Deterministic chaos

4.1 General properties (18 points)

1. Suppose that $f(x)$ is an iterated map that has a stable p-cycle (a cycle of period p) containing the initial point x_0. Show (formally, not mumerically) that the Lyapounov exponent λ is strictly less than 0 (hint: reduce the question to an analysis of the fixed points, make a direct use of the definition of Lyapounov exponents).

2. Show that, if the cycle is superstable, $\lambda = -\infty$.

3. Explain why the following piecewise map $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ has sensitive dependence on the initial conditions (hint: look at the conditions for the stability of a generic point x in the map domain and use the relation between Lyapounov exponents and first order derivatives):

$$f(x) = \begin{cases} 3x & 0 \leq x \leq 1/3 \\ -3x + 2 & 1/3 \leq x \leq 2/3 \\ 3x - 2 & 2/3 \leq x \leq 1 \\ f(x-1) + 1 & x \geq 1 \end{cases} \tag{5}$$

4. Which one of the following statements referring either to a flow or a map would imply the existence of a chaotic behavior?
   (a) The two-dimensional non-linear flow shows sensitive dependence on the initial conditions.
   (b) For values higher than a threshold value of the parameter, the orbits of the one-dimensional map are aperiodic.
   (c) The three dimensional flow shows aperiodic orbits and sensitivity to initial conditions.
   (d) The flow has an attractor with fractal dimension.
   (e) The flow has two positive Lyapounov exponents.

4.2 Tent map (17 points)

Let's consider the so-called tent map, for $0 \leq r \leq 2$, $0 \leq x \leq 1$,

$$f(x) = \begin{cases} rx & 0 \leq x \leq 1/2 \\ r - rx & 1/2 \leq x \leq 1 \end{cases} \tag{6}$$

1. Why do you think is it called the "tent" map? (hint: the answer should be obvious once we plot the graph of the map)

2. Show that the Lyapounov exponent is $\lambda = \log r$ independent of the initial condition x_0 (hint: apply the definition of Lyapounov exponents)

3. The previous result should suggest that the tent map has chaotic behavior for $r > 1$. Construct and plot the orbit diagram as a function of r (you need to write down a software that numerically generates the required points).

4. Construct and plot the value of the Lyapounov exponent as a function of r (here too, you need to write down a software that compute the Lyapounov exponents for different values of r).

5. Discuss the results and identify the regions of chaotic behavior and order (period doubling).