Lecture 10: Dynamical Systems 9

Teacher: Gianni A. Di Caro

15-382 Collective Intelligence – S19
GEOMETRIES IN THE PHASE SPACE

- **Damped pendulum**

\[
\begin{align*}
dx/dt &= y, \\
dy/dt &= -9 \sin x - \frac{1}{5} y
\end{align*}
\]

One cp in the region between two separatrix

**Separatrix**

**Basin of attraction**

- **Saddle**
  - Asymptotically unstable

- **Asymptotically stable spiral (or node)**

**Undamped pendulum**

- **Closed orbits (periodic)**
- **Fixed point (any period)**

Center: the linearization approach doesn’t allow to say much about stability
**Question 1:** The linearization approach for studying the stability of critical points is a purely **local** approach. Going more **global**, what about the **basin of attraction** of a critical point?

**Question 2:** When the *linearization approach fails* as a method to study the stability of a critical point, can we rely on something else?

**Question 3:** Are critical points and *well separated closed orbits* all the **geometries** we can have in the phase space?

**Question 4:** Does the **dimensionality** of the phase space impact on the possible geometries and limiting behavior of the orbits?

**Question 5:** Are critical points and closed orbits the only forms of **attractors** in the dynamics of the phase space? Is *chaos* related to this?
\[ \dot{x} = f(x), \quad f: \mathbb{R}^n \to \mathbb{R}^n \]

\[ V(x(t)) = \text{Potential energy of the system when in state } x, \quad V: \mathbb{R}^n \to \mathbb{R} \]

- Time rate of change of \( V(x(t)) \) along a solution trajectory \( x(t) \), we need to take the derivative of \( V \) with respect to \( t \). Using the chain rule:

\[
\frac{dV}{dt} = \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial V}{\partial x_n} \frac{dx_n}{dt} = \frac{\partial V}{\partial x_1} f_1(x_1, \ldots, x_n) + \cdots + \frac{\partial V}{\partial x_n} f_n(x_1, \ldots, x_n)
\]

*Solutions do not appear, only the system itself!*
LYAPUNOV FUNCTIONS

- \( \dot{x} = f(x), \quad f: \mathbb{R}^n \rightarrow \mathbb{R}^n \)
- \( x^e \) equilibrium point of the system
- A function \( V: \mathbb{R}^n \rightarrow \mathbb{R} \) continuously differentiable is called a **Lyapunov function** for \( x^e \) if for some neighborhood \( D \) of \( x^e \) the following hold:
  1. \( V(x^e) = 0 \), and \( V(x) > 0 \) for all \( x \neq x^e \) in \( D \)
  2. \( \dot{V}(x) \leq 0 \) for all \( x \) in \( D \)
- If \( \dot{V}(x) < 0 \), it’s called a **strict Lyapunov function**

\[ V(x(t)) = \text{Energy of the system when in state } x \]

1. \( x^e \) is the bottom of the graph of the Lyapunov function
2. Solutions can’t move up, but can only move down the side of the potential hole or stay level
**Theorem** (Sufficient conditions for stability):

Let \( x^e \) be an (isolated) equilibrium point of the system \( \dot{x} = f(x) \).

If there exists a Lyapunov function for \( x^e \), then \( x^e \) is stable.

If there exists a strict Lyapunov function for \( x^e \), then \( x^e \) is asymptotically stable.

**Definition:** Let \( x^e \) be an asymptotically stable equilibrium of \( \dot{x} = f(x) \). Then the **basin of attraction** of \( x^e \), denoted \( B(x^e) \), is the set of initial conditions \( x_0 \) such that the associated orbits asymptotically converge to \( x^e \):

\[
\lim_{t \to \infty} F(x_0, t) = x^e
\]
How do we define Lyapunov functions?

- **Physical systems:** Use the energy function of the system itself
  
  For a damped pendulum \((x = \theta, \ y = \frac{d\theta}{dt})\)
  
  \[
  V(x, y) = mgL(1 - \cos x) + \frac{1}{2}mL^2y^2.
  \]

- **Other systems:** Guess!
  
  \[
  \begin{align*}
  \frac{dx}{dt} &= -x - xy^2 \\
  \frac{dy}{dt} &= -y - x^2y
  \end{align*}
  \]

  \[x^e = (0,0)\]

  \[
  V(x, y) = ax^2 + bxy + cy^2
  \]

  The function \(V(x, y) = ax^2 + bxy + cy^2\) is positive definite if, and only if,
  
  \[a > 0 \quad \text{and} \quad 4ac - b^2 > 0,\]

  and is negative definite if, and only if,
  
  \[a < 0 \quad \text{and} \quad 4ac - b^2 > 0.\]

  \[
  \begin{align*}
  V_y(x, y) &= bx + 2cy \\
  V_x(x, y) &= 2ax + by
  \end{align*}
  \]

  \[
  \dot{V}(x, y) = (2ax + by)(-x - xy^2) + (bx + 2cy)(-y - x^2y)
  \]

  \[
  = -[2a(x^2 + x^2y^2) + b(2xy + xy^3 + x^3y) + 2c(y^2 + x^2y^2)].
  \]

  For \(b = 0, \ a, c > 0\) \(\rightarrow \dot{V} < 0, \ V > 0\) \(\Rightarrow (0,0)\) is asymptotically stable