15-382 Collective Intelligence – S19

Lecture 14:
Discrete-Time Dynamical Systems 2

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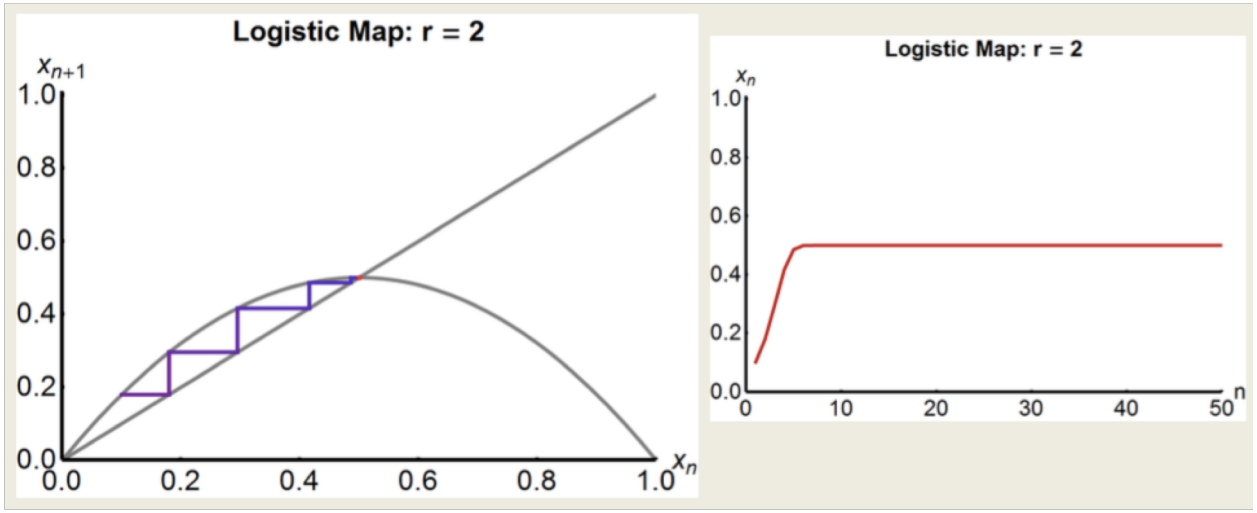
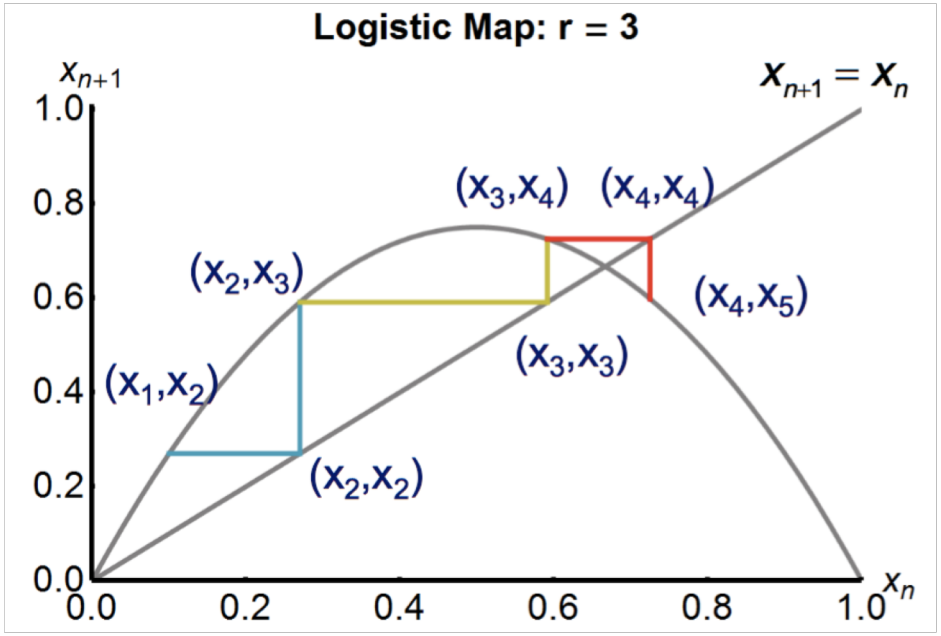
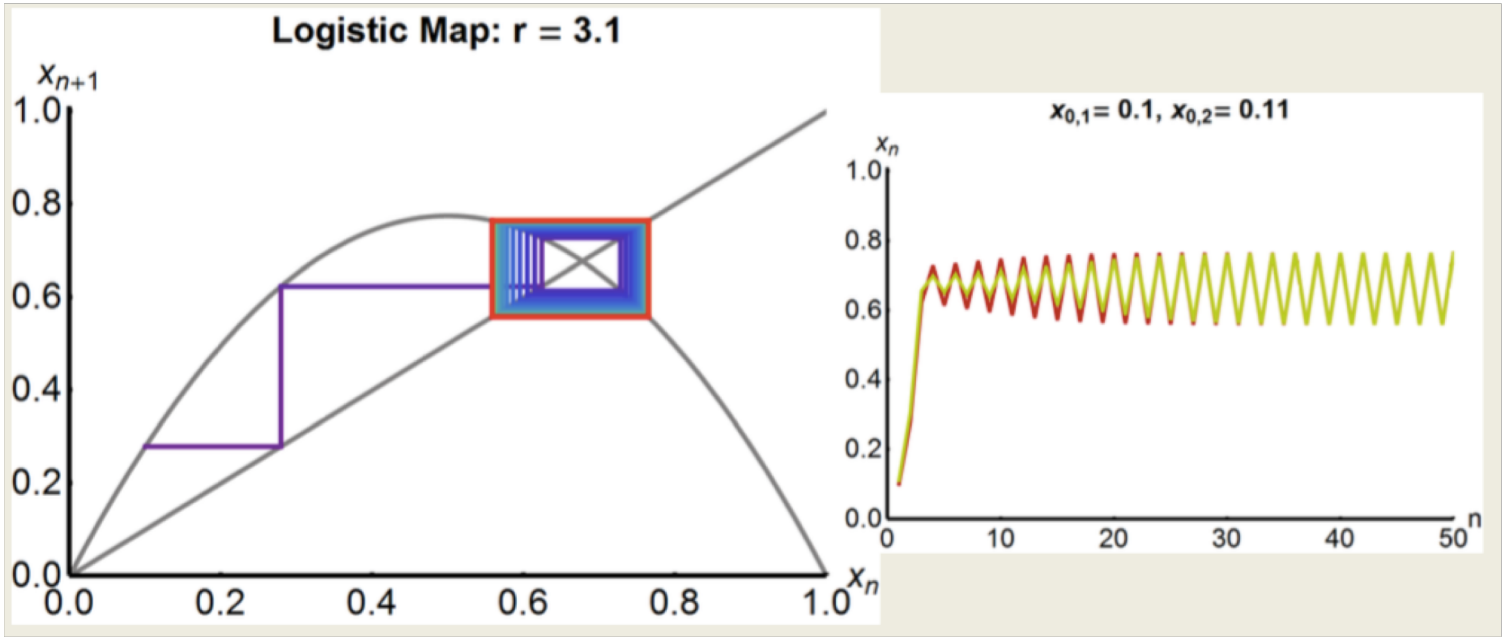
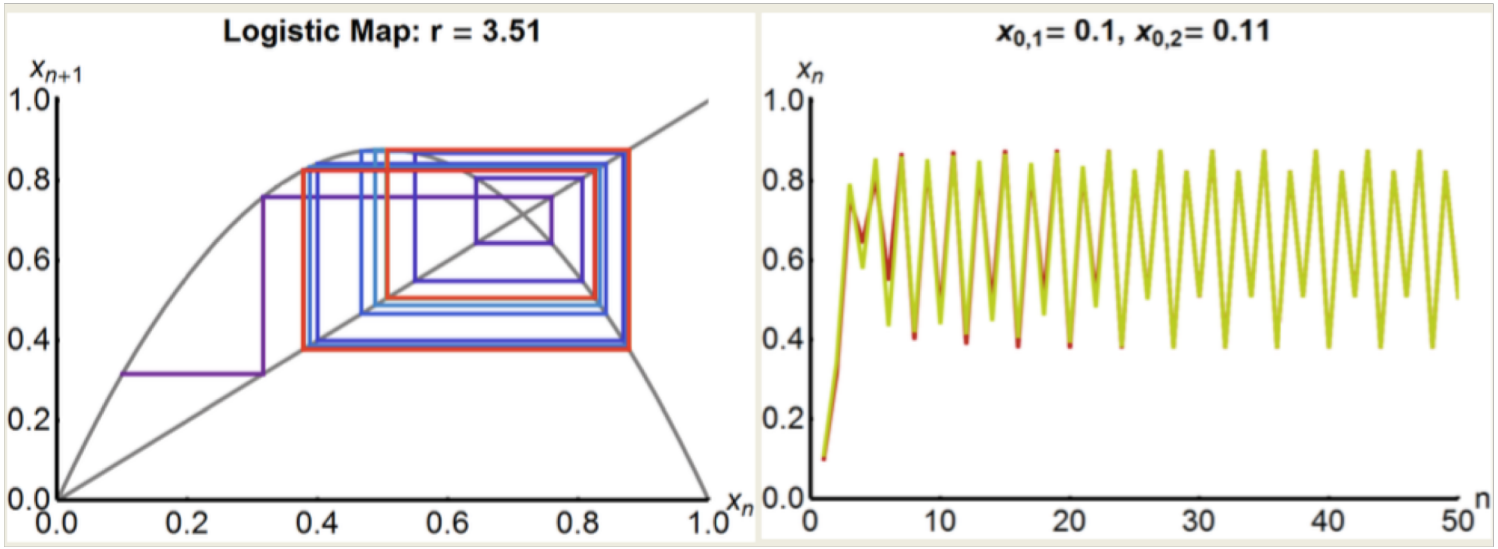
Regular behavior, periodic attractors



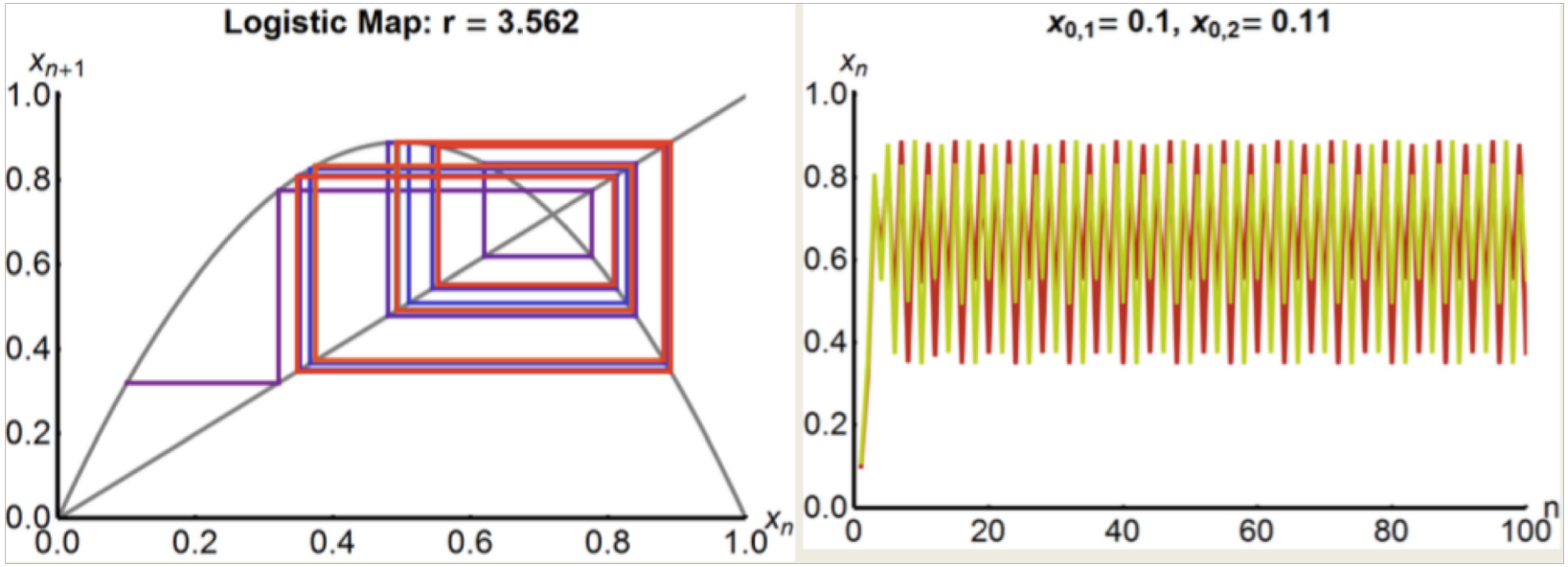


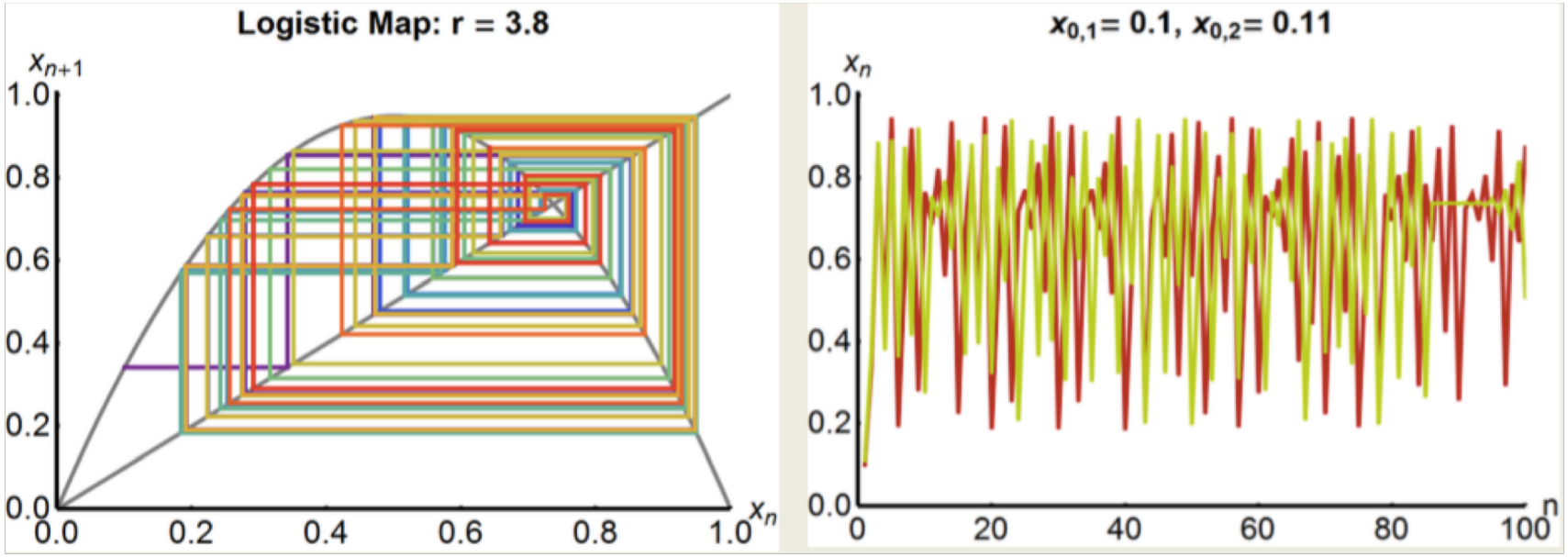
REGULAR BEHAVIOR, PERIODIC ATTRACTORS



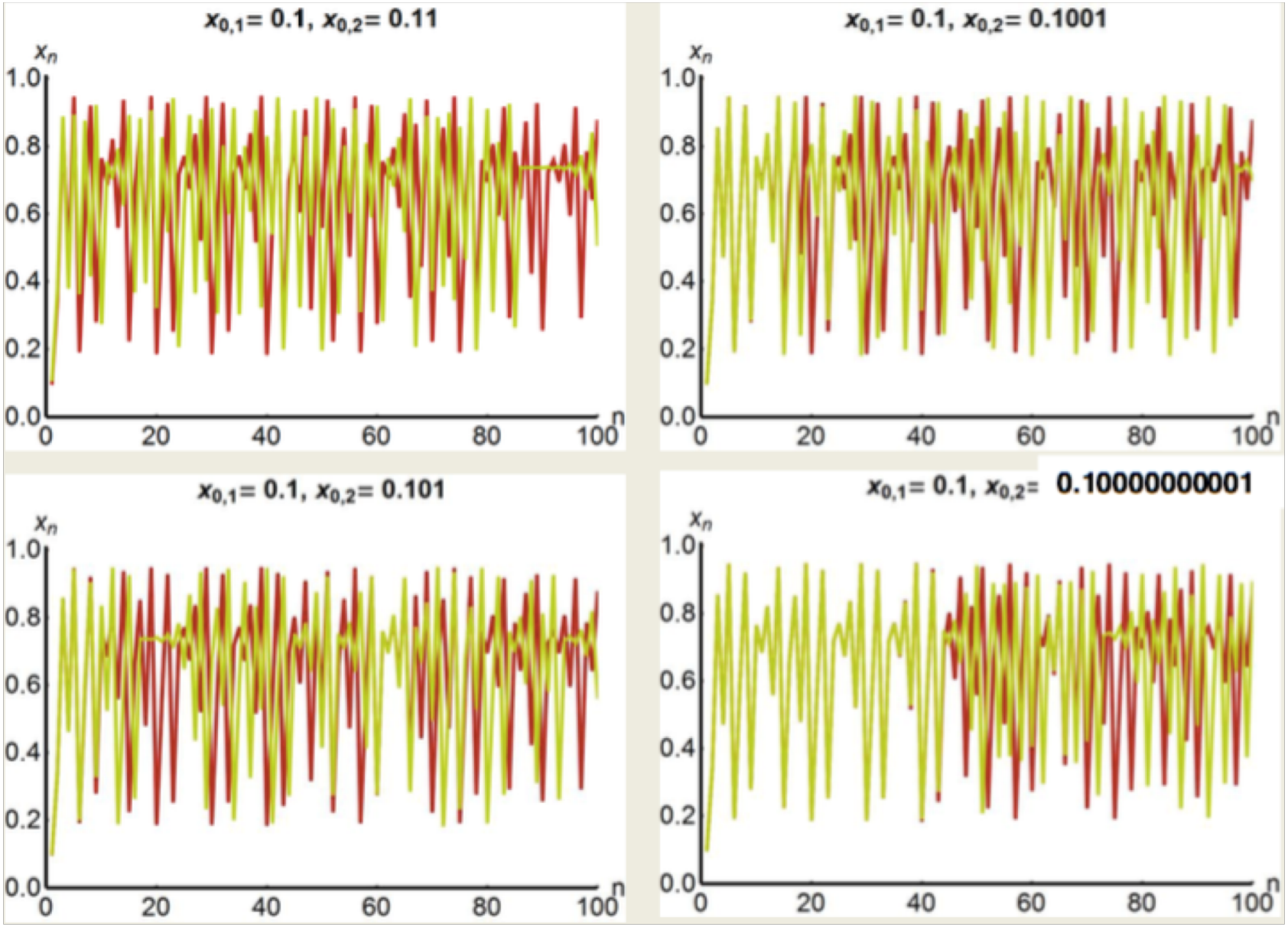


TRANSITION TO CHAOTIC BEHAVIOR



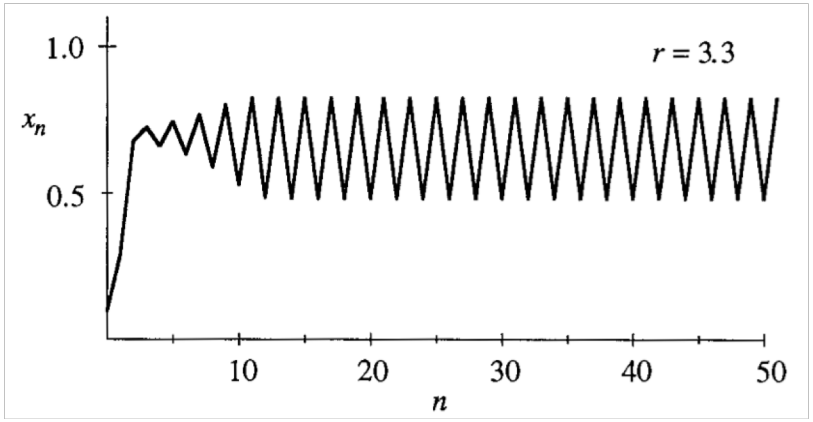


CHAOS: SENSITIVITY TO INITIAL CONDITIONS

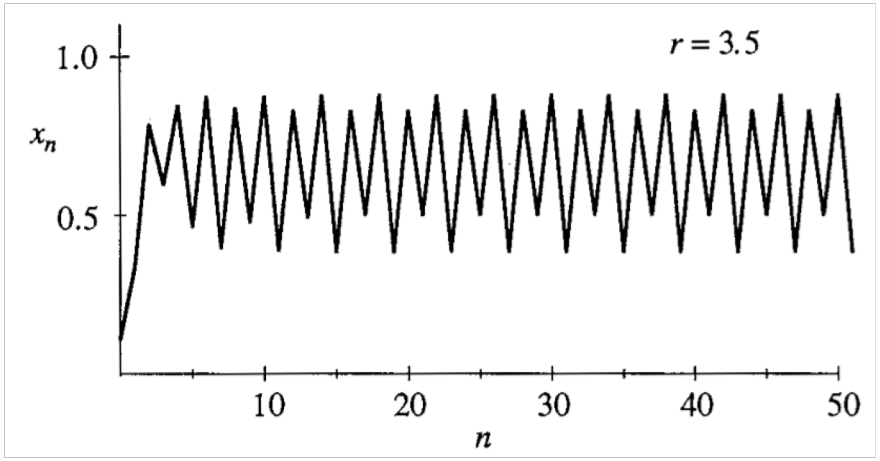


PERIODS IN THE LOGISTIC MAP

- *Oscillating* about the previous steady state, alternating between small and large populations
- **Period-2 cycle:** Oscillation repeats every two iterations, <u>periodic orbit</u>



- **Period-4 cycle**



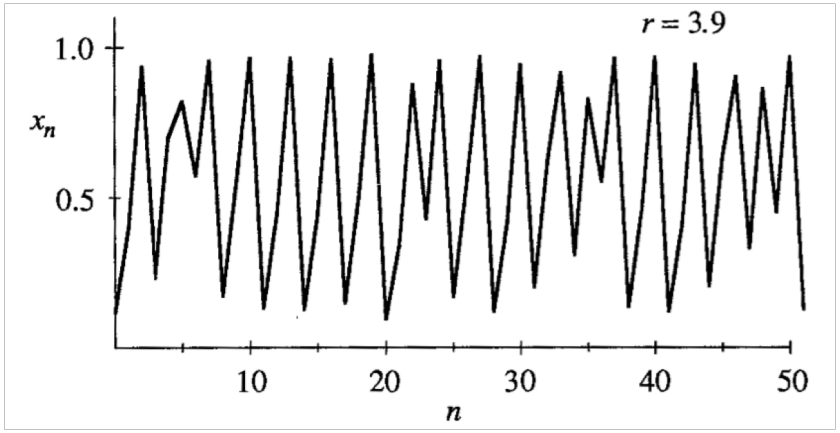
$r_1 = 3$	(period 2 is born)
$r_2 = 3.449\ldots$	4
$r_3 = 3.54409\ldots$	8
$r_4 = 3.5644\ldots$	16
$r_5 = 3.568759\ldots$	32
$\vdots$	$\vdots$
$r_\infty = 3.569946\ldots$	∞

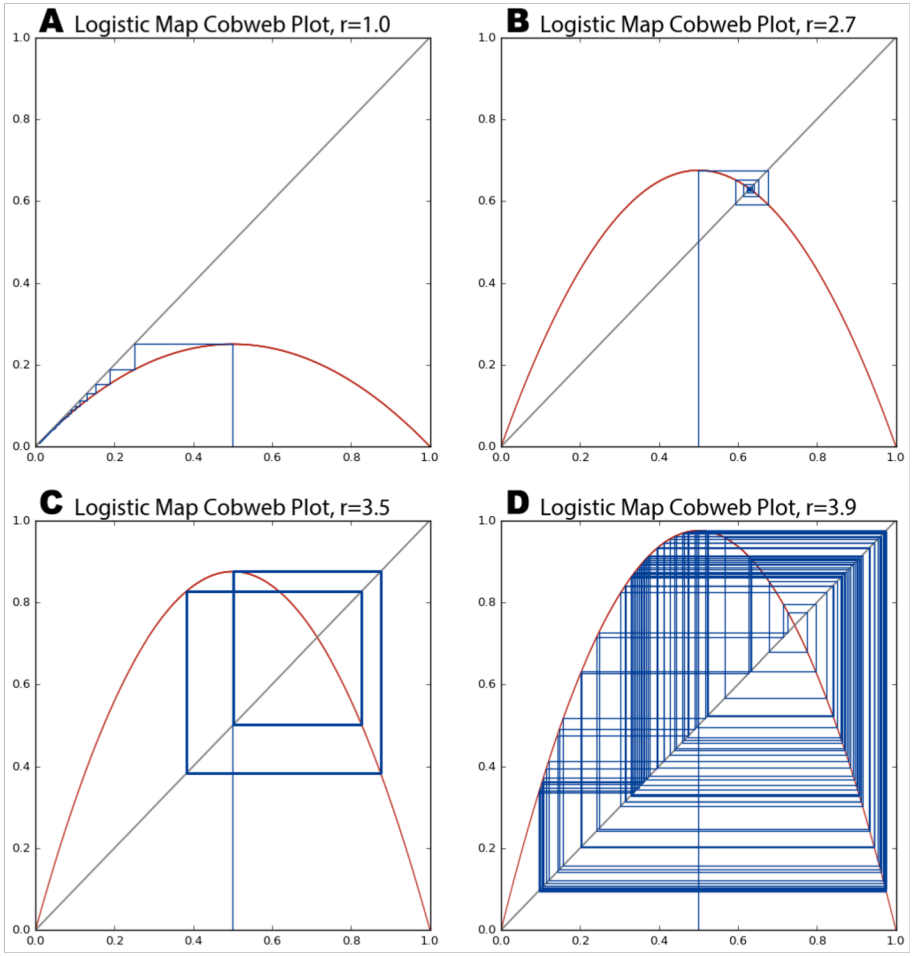
- **Period-doubling** to cycles appears by increasing r
- They correspond to **bifurcations i**n phase diagram
- Successive bifurcations come faster and faster!
- Limiting value $r_n \to r_\infty = 3.569946\ldots$
- <u>Geometric convergence</u>, in the limit the distance between successive values shrink to a constant:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

CHAOS …

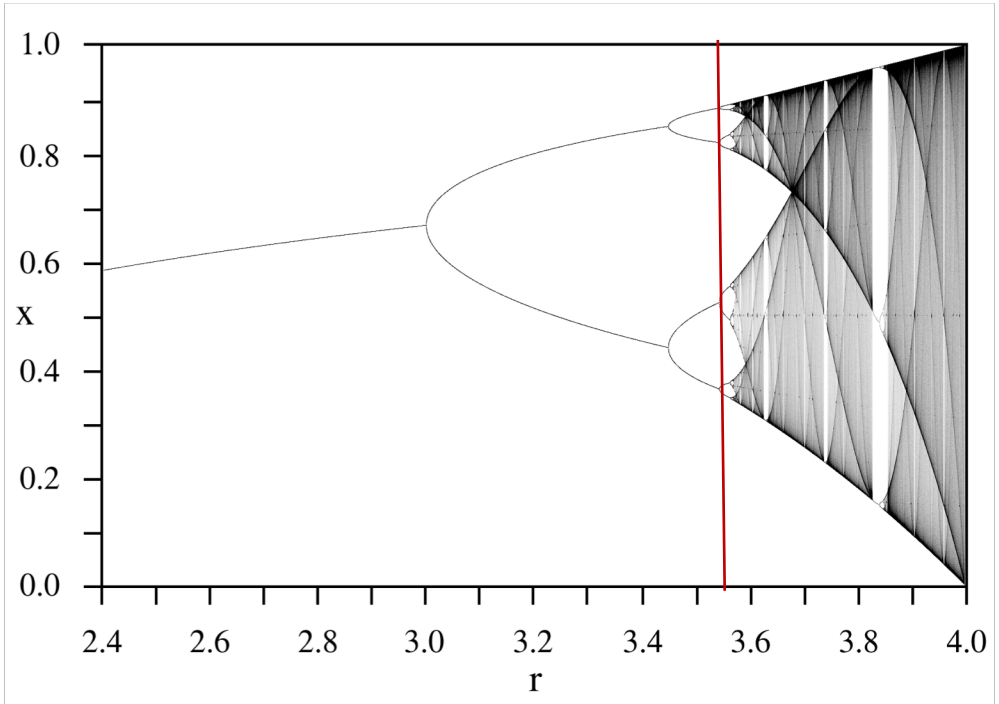
- $r > r_\infty$?
- For many values of r, the sequence never settles down to a fixed point or a periodic orbit
- **Aperiodic, bounded behavior!**





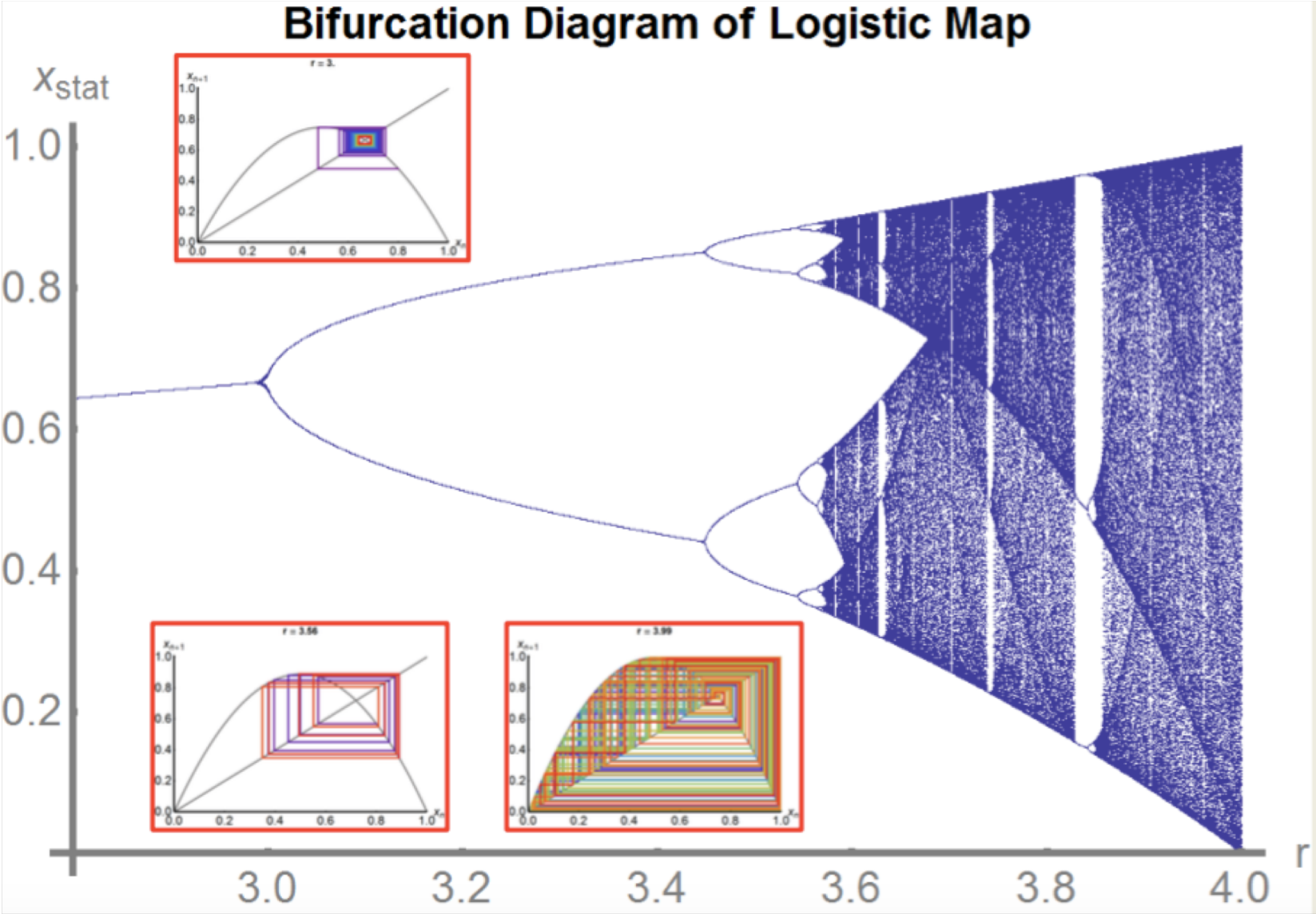
ORBIT DIAGRAM

- What happens for larger r? Sure, more *chaos*…. Even more interesting things!
- **Orbit diagram:** system's attractors as a function of r
- *Construction:*
  - Choose a value of r
  - Select a random initial condition x_0 and generate the orbit: lets iterate for ~300 cycles to let the system settle down, then plot the next ~300 points from the map iterations
  - Move to an adjacent value of r and repeat, sweeping the r interval

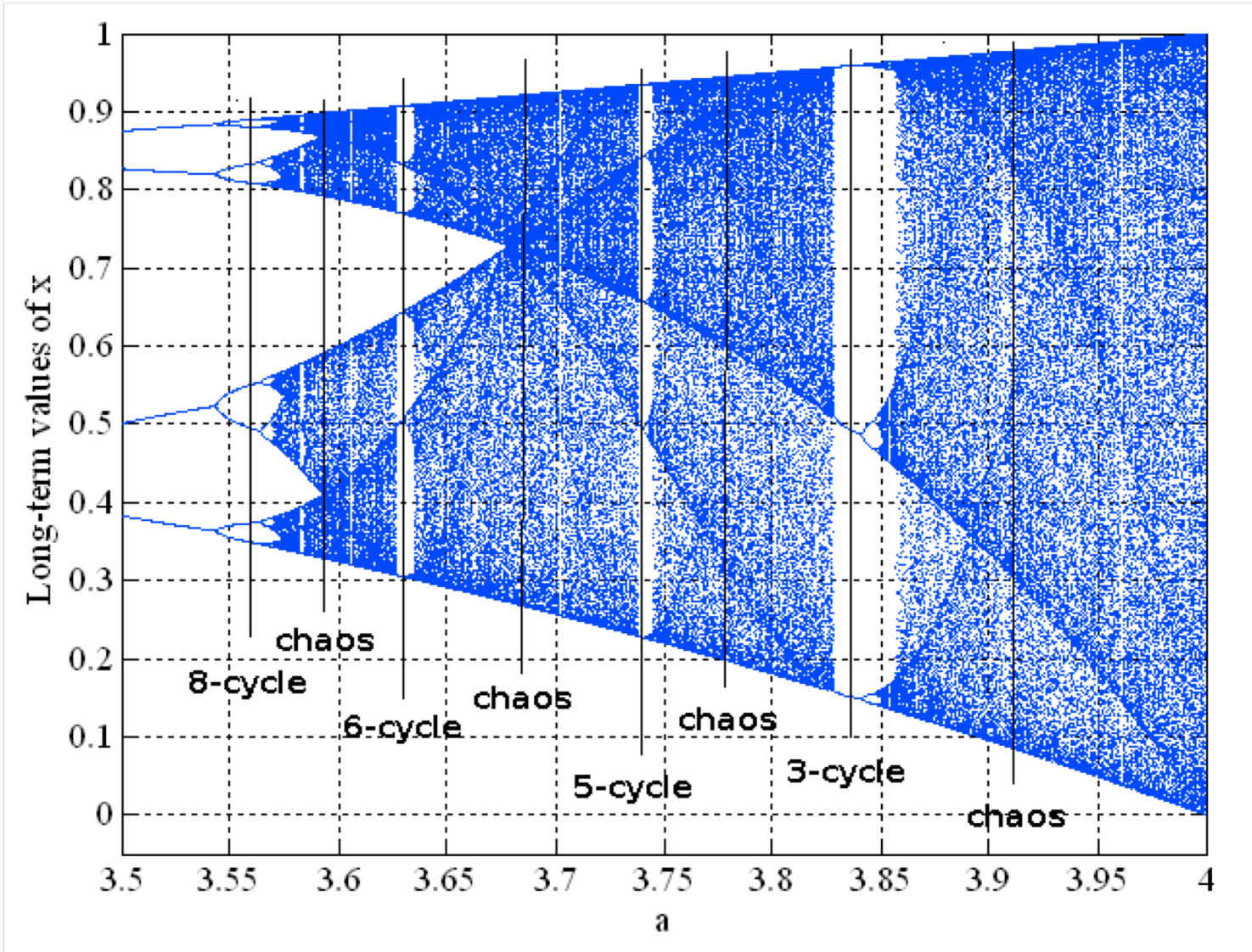


- At $r \approx r_\infty = 3.57$ the map becomes chaotic and the attractor changes from a finite to an infinite set of points
- For $r > 3.57$, mixture of order and chaos, with periodic windows interspersed between clouds of (chaotic) dots

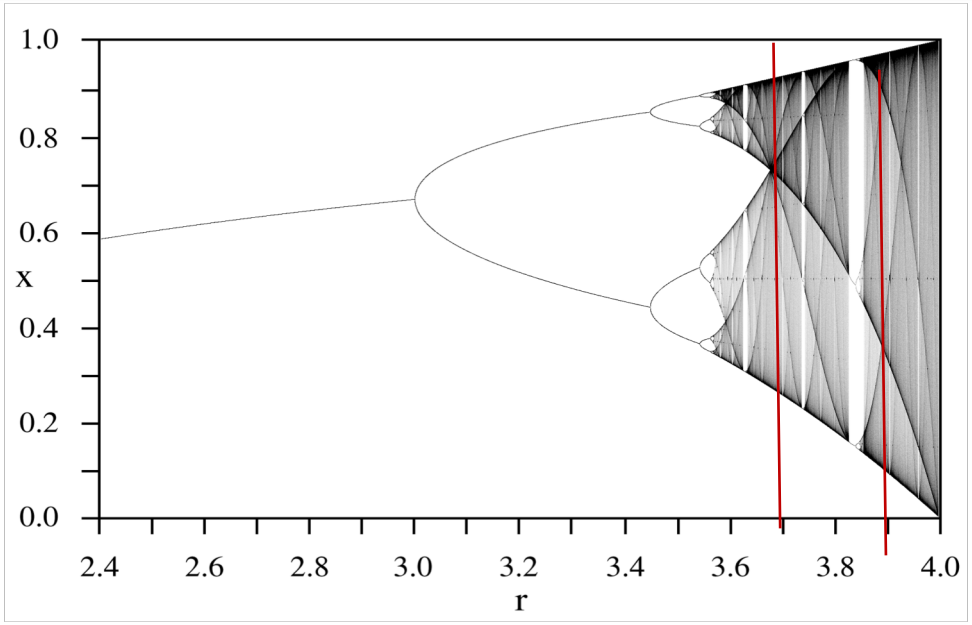
ORBIT DIAGRAM



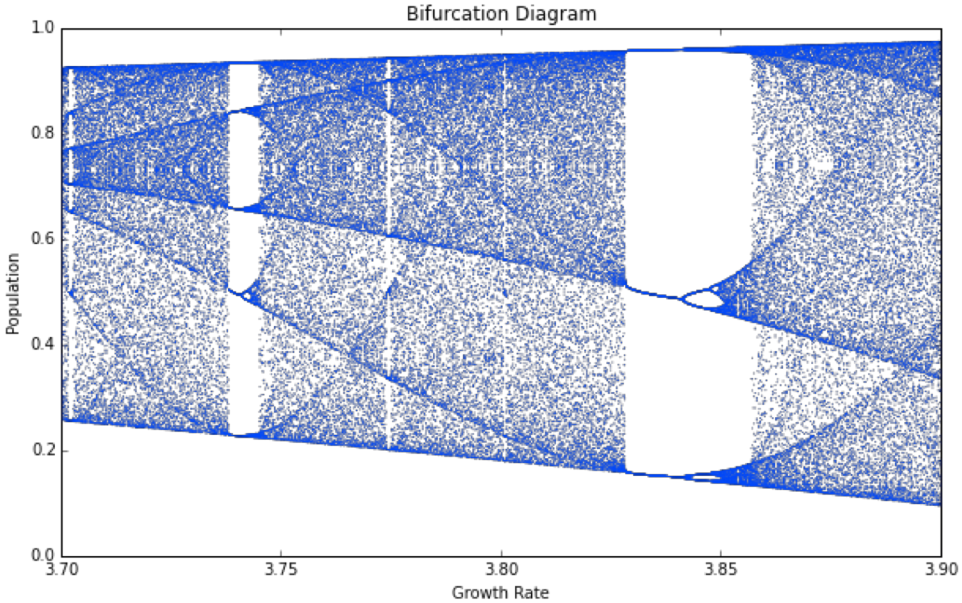
CHAOS AND ORDER

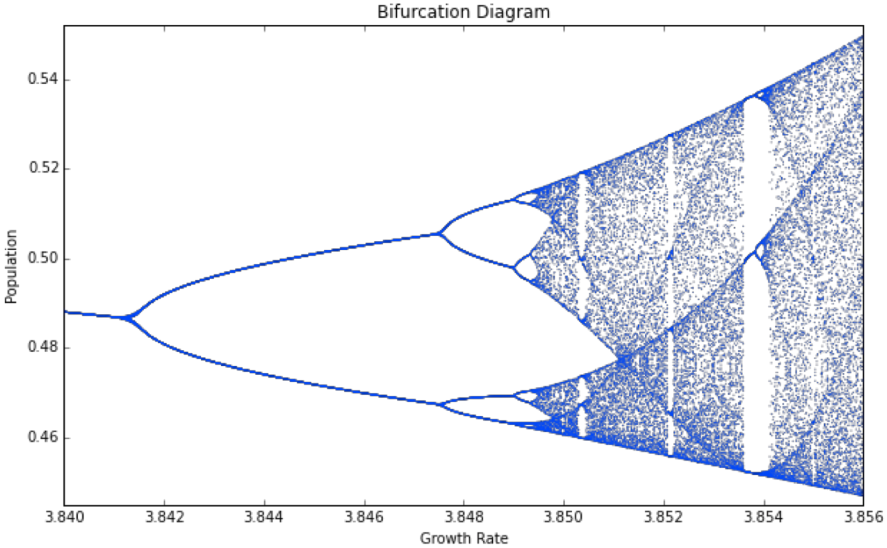


Chaos and order, Self-similarity



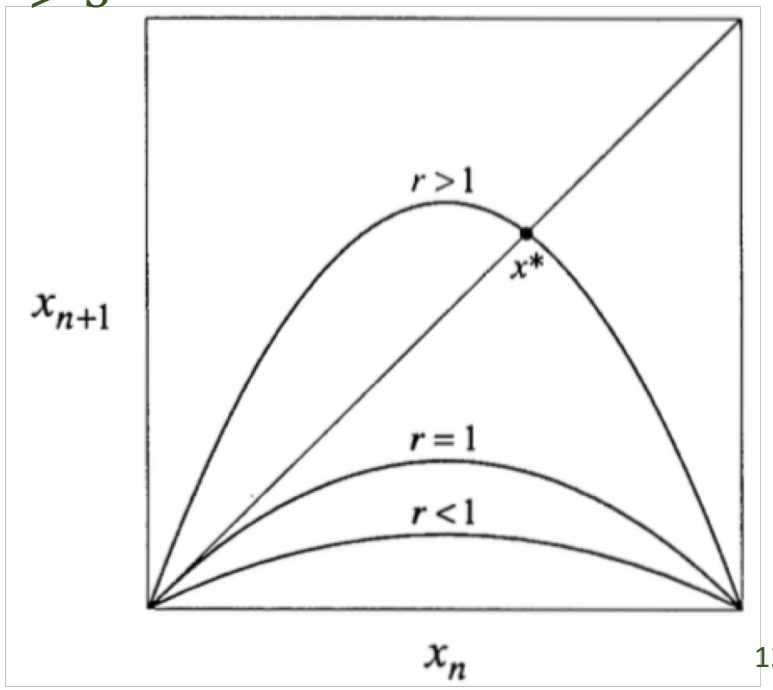
- The large window at $r \approx 3.83$ contains a stable period-3 orbit
- Looking at the period-3 window even closer: a copy of the orbit diagram reappears in miniature! → **Self-similarity**





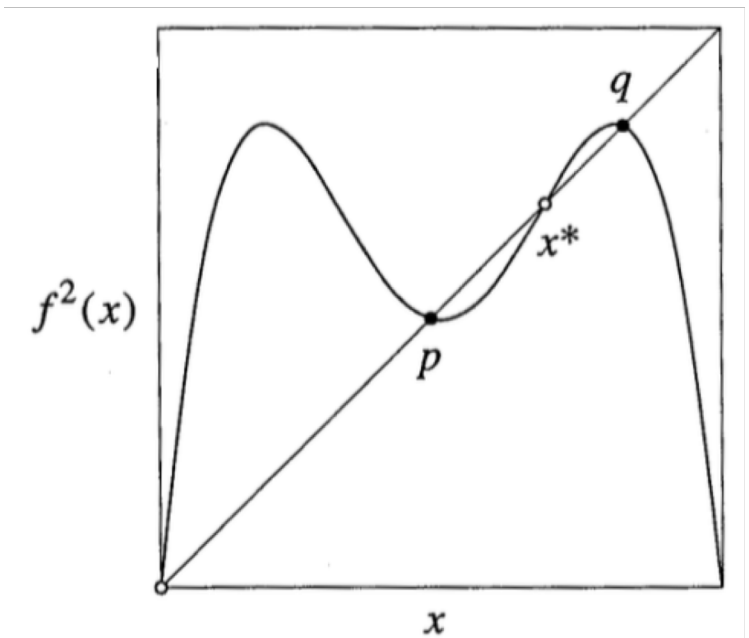
LOGISTIC MAP: ANALYSIS

- $x_{n+1} = rx_n(1 - x_n),\ 0 \le r \le 4,\ 0 \le x \le 1$, Fixed points and stability?
- Fixed points, are roots of: $x^* = f(x^*) = rx^*(1 - x^*) \rightarrow x^* = 0,\ x^* = 1 - \frac{1}{r}$
- Since $x \ge 0, x^*$ is in the range of allowable values only if $r \ge 1$
- Stability depends on multiplier $\lambda = f'(x^*) = r - 2rx^*$
- $x^* = 0$: $f'(x^* = 0) = r \rightarrow$ Origin is stable for $r < 1$, unstable for $r > 1$
- $x^* = 1 - \frac{1}{r}$: $f'\left(1 - \frac{1}{r}\right) = 2 - r \rightarrow$
  $1 - \frac{1}{r}$ is stable for $1 < r < 3$, unstable for $r > 3$
- For $r = 1$, a second fixed point appears, while the origin loses its stability
- $\rightarrow$ **Transcritical bifurcation** *at* $r = 1$
- When the slope of the parabola at $x^* = 0$ becomes too steep, the origin loses its stability (it happens at $r = 1$)
- $\rightarrow$ **Flip bifurcation** at $r = 3$, that are (usually) associated with period doubling and in this case a *2-period cycle is spawn*



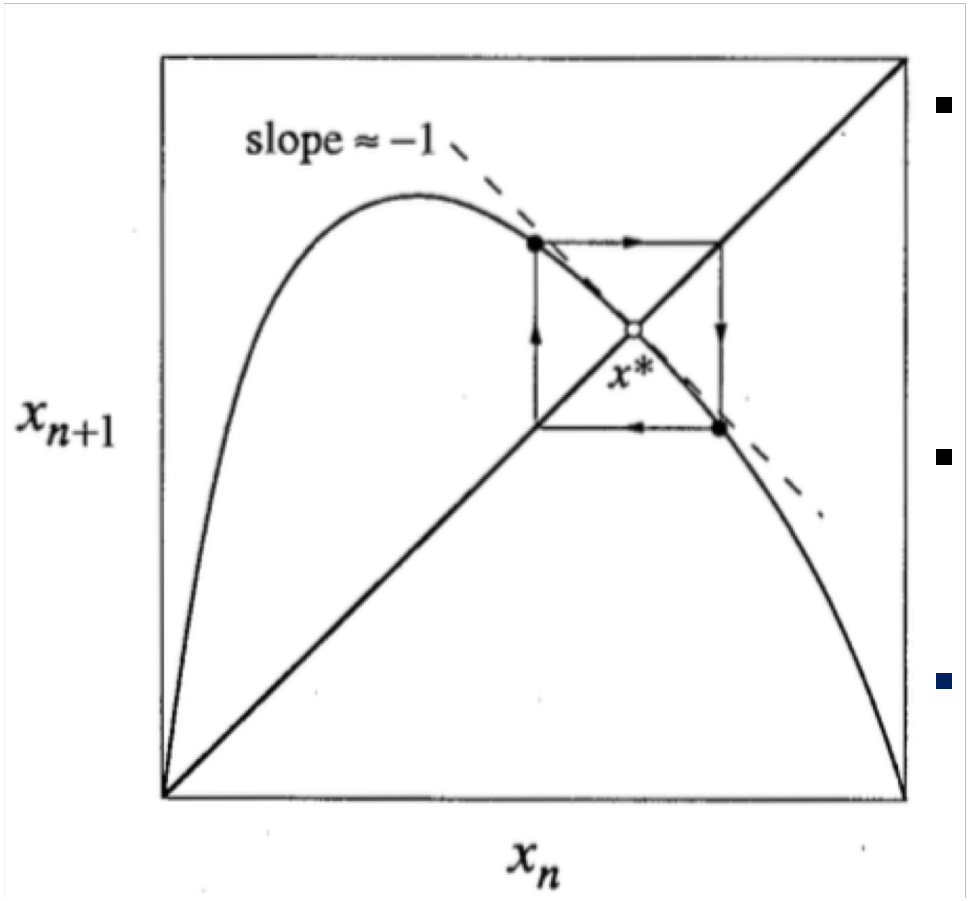
ANALYSIS: SPAWNING OF TWO-CYCLE

- The logistic map has a *two-cycle* for all $r > 3$
- Period-2 cycle: there are two states p and q, such that :
  - $f(p) = q,\ f(q) = p$, or equivalently,
  - $f\big(f(p)\big) = p$
  - → p (and q) fixed points of second-iterate map, $f^2(x) \equiv f\big(f(x)\big)$
- $f^2(x)$ is a quartic polynomial, that for $r > 3$ looks like:



- p, q corresponds to where the graph of $f^2(x)$ intersects the diagonal: $f^2(x) = x$
- … $p, q = \frac{r+1\sqrt{(r-3)(r+1)}}{2r}$, real for $r > 3$
- → A two-cycle exists for all $r > 3$
- At $r = 3$, the two-cycle *bifurcates* continuously from x^*

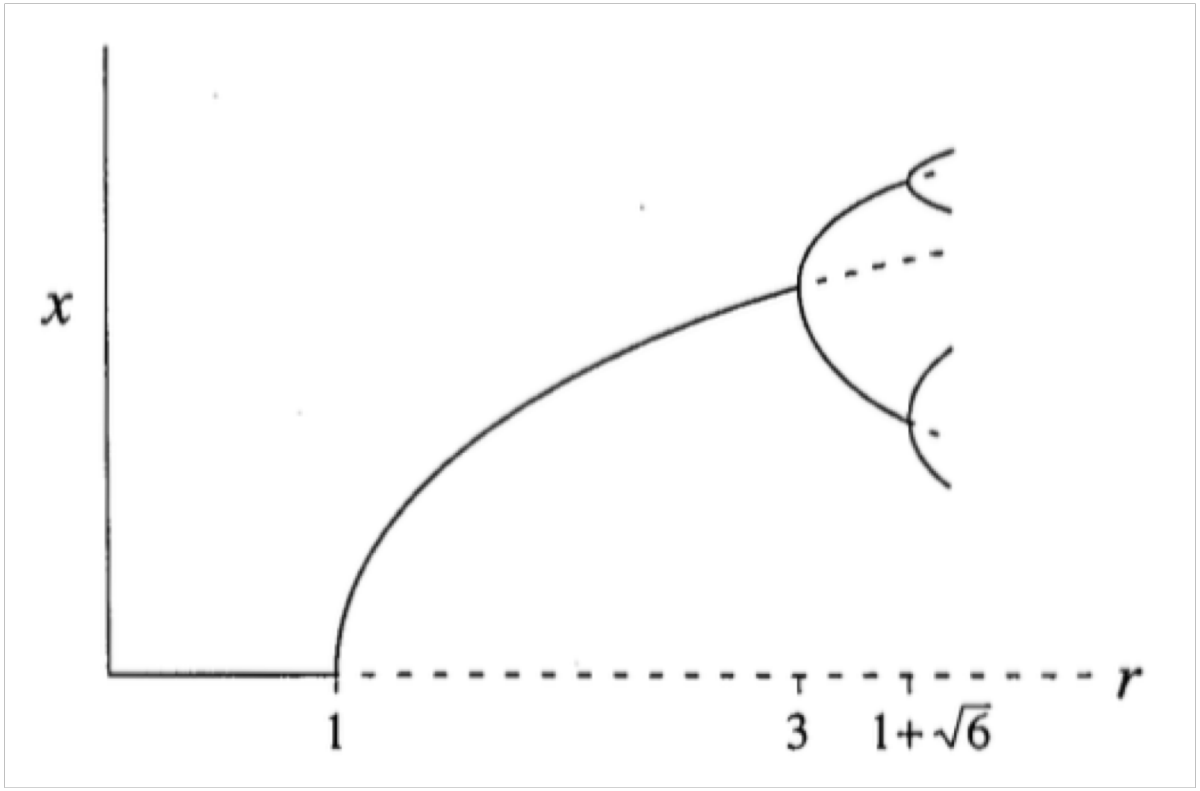
FLIP BIFURCATIONS AND PERIOD DOUBLING



- If tangent slope $f'(x^*) \approx -1$ and the graph of the function is *concave* near x^*, the cobweb tends to produce a small, stable 2-cycle around the fixed point
- The critical slope ≈ -1 corresponds to a **flip bifurcation** that gives rise two a 2-cycle
- How can we determine that the 2-cycle is stable or not?

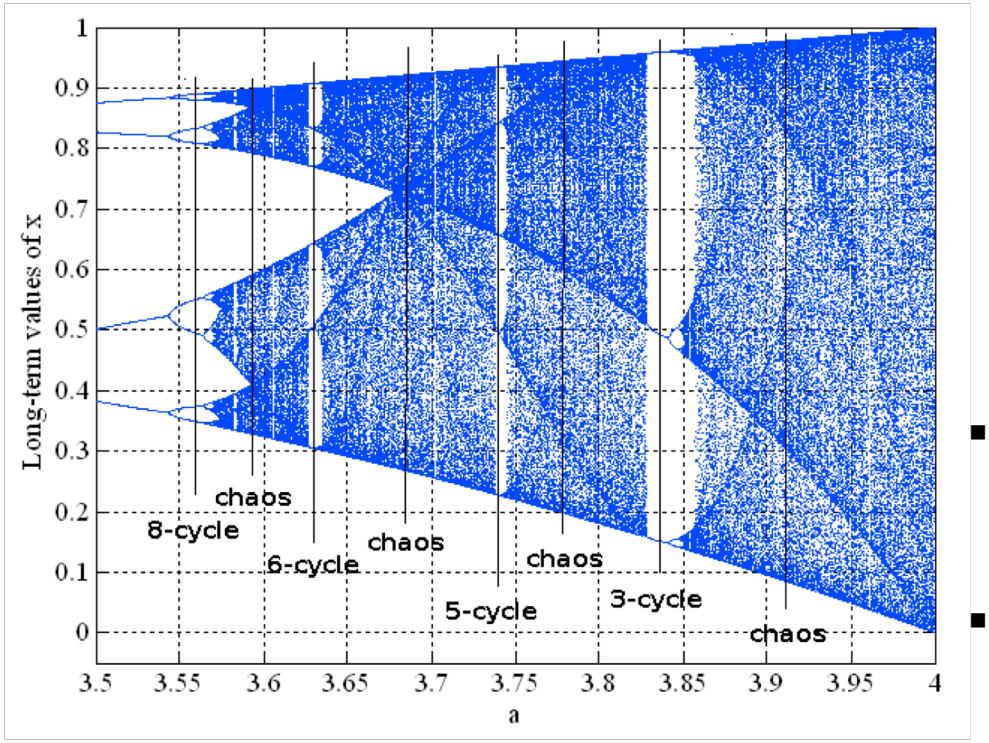
- p, q are the solutions of $f^2(x) = x$ → The 2-cycle determined by p, q is stable iff p, q are both stable fixed points of the f^2 map
- Doing the usual analysis … for both p, q $\rightarrow \lambda = 4 - 2r - r^2$
- → The 2-cycle is stable iff $|4 - 2r - r^2| < 1$ → $r < 1 + \sqrt{6}$

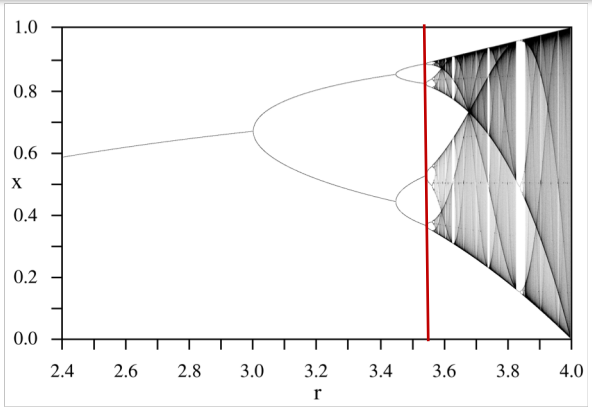
A FIRST BIFURCATION DIAGRAM ...



- The dashed lines indicate fixed points that are instable
- The first bifurcation is a <u>flip one</u>, that creates a new equilibrium, losing the stability of the original one
- Each further <u>pitchfork bifurcation</u> is a <u>supercritical</u> one, with two new stable equilibrium points appearing and the original equilibrium losing its stability.

Occurrence of periodic windows for $r > r_\infty$

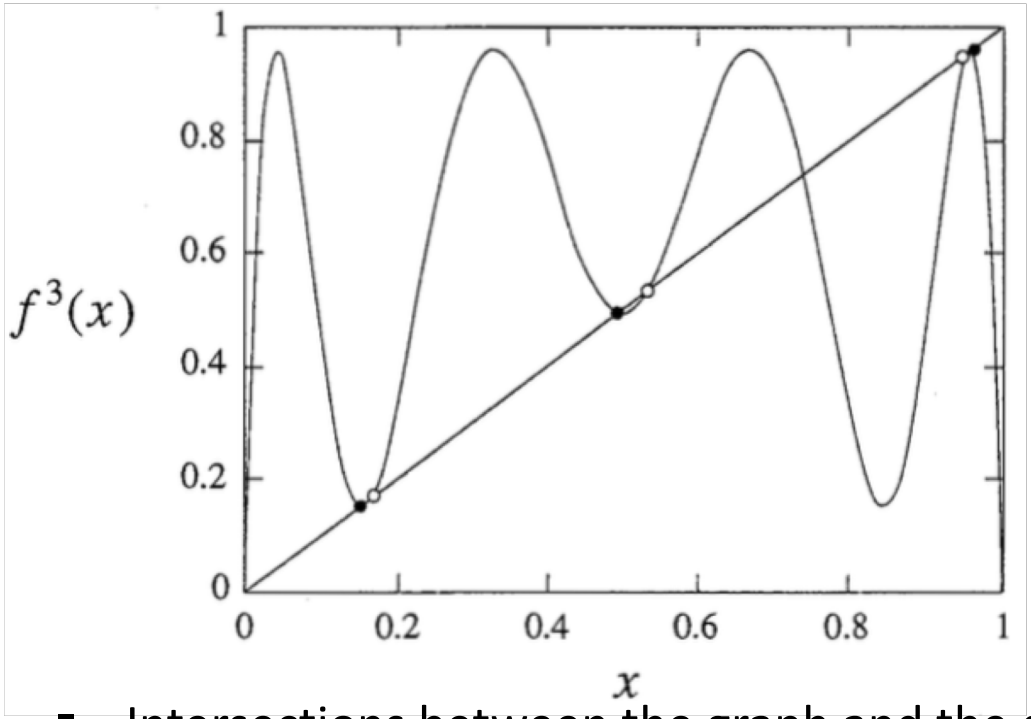




- At $r \approx r_\infty = 3.57$ the map becomes chaotic and the attractor changes from a finite to an infinite set of points
- The large window at $r \approx 3.83$ contains a stable period-3 orbit

- $f(x) = rx(1-x)$ → the logistic map is $x_{n+1} = f(x_n)$
- $x_{n+2} = f\big(f(x_n)\big), \quad x_{n+3} = f(f(f(x_n))) = f^3(x_n)$
- We are looking for 3-period cycles: every point p in a 3-period cycle repeats every 3 iterates
- → p must satisfy $p = f^3(p)$ → p is a fixed-point of the f^3 map
- Unfortunately, the f^3 map is an 8-degree polynomial, a bit complex to study

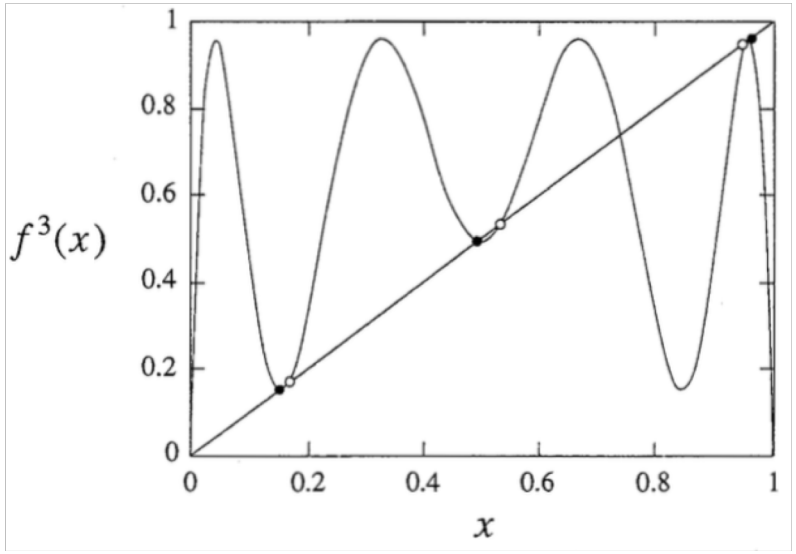
Occurrence of periodic windows for $r > r_\infty$



$r = 3.835$, inside 3-period window

- Intersections between the graph and the diagonal correspond to the solutions of $f^3(x) = x$
- Only the black dots correspond to fixed points, and there are 3 of them, corresponding to the the 3-period cycle
- The slope of the function, $|f'|$ is greater than 1 for the white dots, and less than 1 for the black ones
- For the other intersections, they correspond to fixed points or 1-period

Occurrence of periodic windows for $r > r_\infty$



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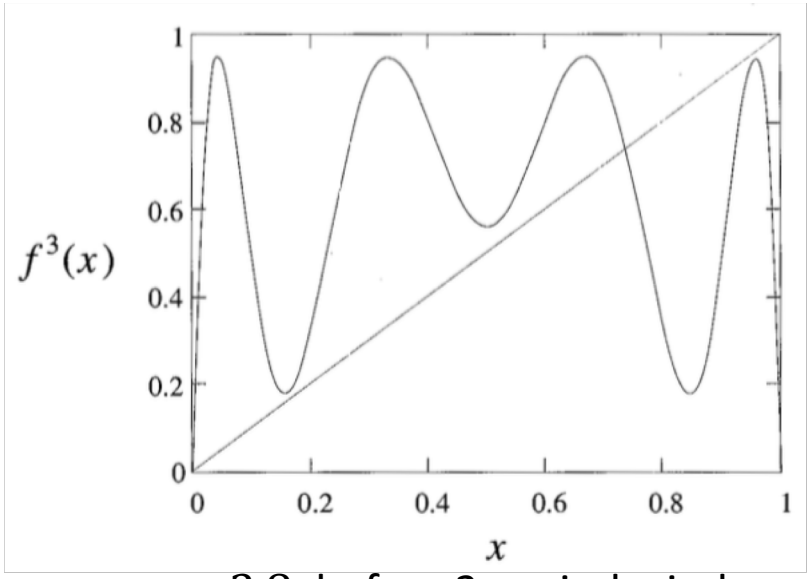
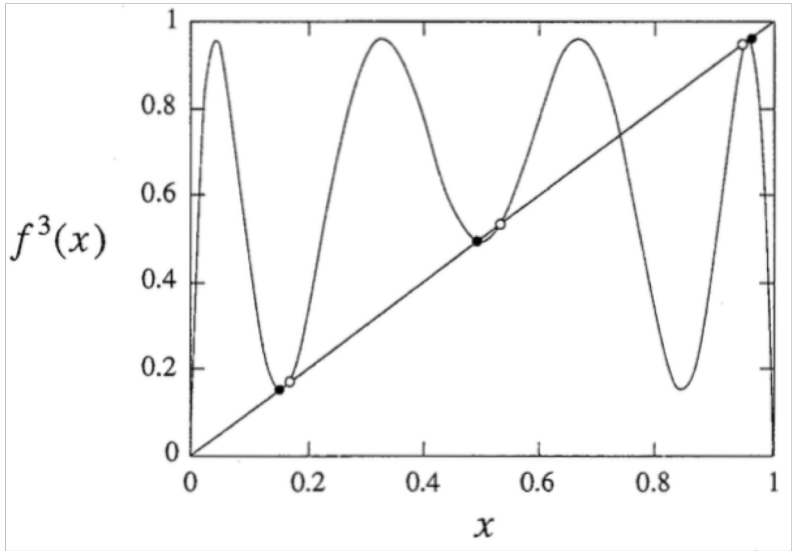


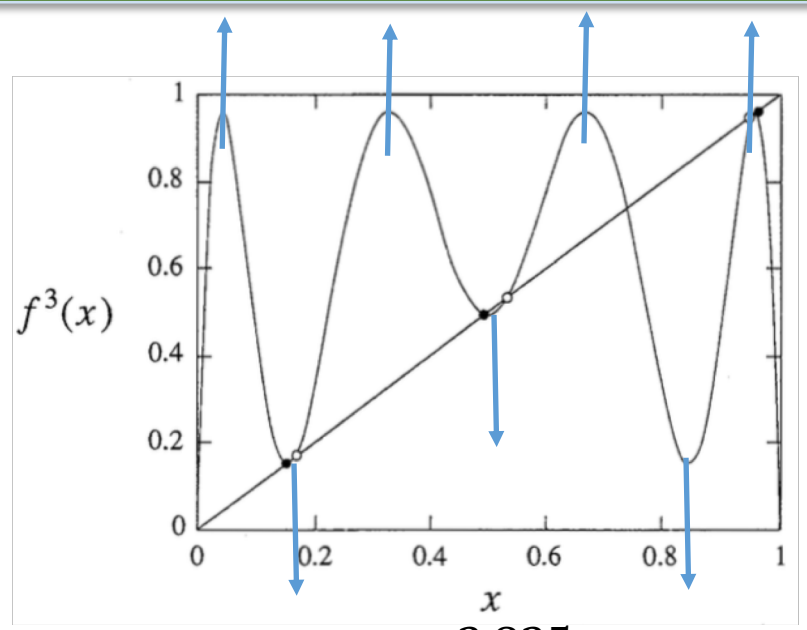
$r = 3.8$, before 3-period window

- The 6 intersections of interest have vanished!
- Not anymore periodic behavior
- For *some* r between 3.8 and 3.835 the graph is <u>tangent</u> to the diagonal
- At this critical value of r, the stable and unstable 3-period cycles coalesce and annihilate in a **tangent bifurcation**, that sets the beginning of the periodic window
- It can be computed analytically that this happens at $r = 1 + \sqrt{6}$

Occurrence of periodic windows for $r > r_\infty$



$r = 3.835$, inside 3-period window



$r > 3.835$

- Just after the tangent bifurcation, the slope at black dots (periodic points) is $\approx +1$ (a bit less)
- For increasing values of r, hills and valleys become steeper / deeper
- The slope of f^3 at the black dots decreases steadily from $\approx +1$ to -1. When this occurs, a **flip bifurcation** happens, that causes each of the fixed periodic points to <u>split in two</u>
- → <u>the 3-period cycle becomes a *6-period cycle*!</u>
- … the process iterates as the map iterates, bringing the <u>period doubling cascade!</u>