LECTURE 14: DISCRETE-TIME DYNAMICAL SYSTEMS 2

TEACHER: GIANNI A. DI CARO
Regular behavior, periodic attractors
REGULAR BEHAVIOR, PERIODIC ATTRACTORS
TRANSITION TO CHAOTIC BEHAVIOR
CHAOS: SENSITIVITY TO INITIAL CONDITIONS
**PERIODS IN THE LOGISTIC MAP**

- *Oscillating* about the previous steady state, alternating between small and large populations

- **Period-2 cycle**: Oscillation repeats every two iterations, periodic orbit

- **Period-doubling** to cycles appears by increasing $r$
  - They correspond to **bifurcations** in phase diagram
  - Successive bifurcations come faster and faster!
  - Limiting value $r_n \to r_\infty = 3.569946...$
  - Geometric convergence, in the limit the distance between successive values shrink to a constant:

\[
\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669...\]
- \( r > r_{\infty} \) ?
- For many values of \( r \), the sequence never settles down to a fixed point or a periodic orbit
- Aperiodic, bounded behavior!
What happens for larger $r$? Sure, more chaos.... Even more interesting things!

**Orbit diagram:** system’s attractors as a function of $r$

**Construction:**
- Choose a value of $r$
- Select a random initial condition $x_0$ and generate the orbit: let's iterate for $\sim300$ cycles to let the system settle down, then plot the next $\sim300$ points from the map iterations
- Move to an adjacent value of $r$ and repeat, sweeping the $r$ interval

At $r \approx r_\infty = 3.57$ the map becomes chaotic and the attractor changes from a finite to an infinite set of points

For $r > 3.57$, mixture of order and chaos, with periodic windows interspersed between clouds of (chaotic) dots
ORBIT DIAGRAM

Bifurcation Diagram of Logistic Map

\[ x_{n+1} = rx_n(1-x_n) \]
CHAOS AND ORDER
CHAOS AND ORDER, SELF-SIMILARITY

- The large window at $r \approx 3.83$ contains a stable period-3 orbit
- Looking at the period-3 window even closer: a copy of the orbit diagram reappears in miniature! → Self-similarity
Logistic Map: Analysis

- $x_{n+1} = rx_n(1 - x_n), \ 0 \leq r \leq 4, \ 0 \leq x \leq 1$, Fixed points and stability?

- Fixed points, are roots of: $x^* = f(x^*) = rx^*(1 - x^*) \Rightarrow x^* = 0, \ x^* = 1 - \frac{1}{r}$

- Since $x \geq 0$, $x^*$ is in the range of allowable values only if $r \geq 1$

- Stability depends on multiplier $\lambda = f'(x^*) = r - 2rx^*$

- $x^* = 0$: $f'(x^* = 0) = r \Rightarrow$ Origin is stable for $r < 1$, unstable for $r > 1$

- $x^* = 1 - \frac{1}{r}$: $f' \left(1 - \frac{1}{r}\right) = 2 - r \Rightarrow$

  $1 - \frac{1}{r}$ is stable for $1 < r < 3$, unstable for $r > 3$

- For $r = 1$, a second fixed point appears, while the origin loses its stability

  $\Rightarrow$ **Transcritical bifurcation at** $r = 1$

- When the slope of the parabola at $x^* = 0$ becomes too steep, the origin loses its stability (it happens at $r = 1$)

  $\Rightarrow$ **Flip bifurcation at** $r = 3$, that are (usually) associated with period doubling and in this case a **2-period cycle is spawn**
The logistic map has a two-cycle for all $r > 3$

Period-2 cycle: there are two states $p$ and $q$, such that:

- $f(p) = q$, $f(q) = p$, or equivalently,
- $f(f(p)) = p$
- $\Rightarrow p$ (and $q$) fixed points of second-iterate map, $f^2(x) \equiv f(f(x))$

$f^2(x)$ is a quartic polynomial, that for $r > 3$ looks like:

- $p, q$ corresponds to where the graph of $f^2(x)$ intersects the diagonal: $f^2(x) = x$
- $... \ p, q = \frac{r+1\sqrt{(r-3)(r+1)}}{2r}$, real for $r > 3$
- $\Rightarrow$ A two-cycle exists for all $r > 3$
- At $r = 3$, the two-cycle bifurcates continuously from $x^*$
Flip bifurcations and period doubling

- If tangent slope \( f'(x^*) \approx -1 \) and the graph of the function is concave near \( x^* \), the cobweb tends to produce a small, stable 2-cycle around the fixed point
- The critical slope \( \approx -1 \) corresponds to a flip bifurcation that gives rise to a 2-cycle
- How can we determine that the 2-cycle is stable or not?

- \( p, q \) are the solutions of \( f^2(x) = x \) \( \rightarrow \) The 2-cycle determined by \( p, q \) is stable iff \( p, q \) are both stable fixed points of the \( f^2 \) map
- Doing the usual analysis ... for both \( p, q \) \( \rightarrow \) \( \lambda = 4 - 2r - r^2 \)
- \( \rightarrow \) The 2-cycle is stable iff \( |4 - 2r - r^2| < 1 \) \( \rightarrow \) \( r < 1 + \sqrt{6} \)
The dashed lines indicate fixed points that are instable

The first bifurcation is a \textit{flip} one, that creates a new equilibrium, losing the stability of the original one

Each further \textit{pitchfork bifurcation} is a \textit{supercritical} one, with two new stable equilibrium points appearing and the original equilibrium losing its stability.
Occurrence of periodic windows for \( r > r_\infty \)

- At \( r \approx r_\infty = 3.57 \) the map becomes chaotic and the attractor changes from a finite to an infinite set of points.
- The large window at \( r \approx 3.83 \) contains a stable period-3 orbit.

- \( f(x) = rx(1 - x) \) \( \Rightarrow \) the logistic map is \( x_{n+1} = f(x_n) \)
- \( x_{n+2} = f(f(x_n)), \quad x_{n+3} = f(f(f(x_n))) = f^3(x_n) \)
- We are looking for 3-period cycles: every point \( p \) in a 3-period cycle repeats every 3 iterates
- \( \rightarrow p \) must satisfy \( p = f^3(p) \) \( \Rightarrow p \) is a fixed-point of the \( f^3 \) map
- Unfortunately, the \( f^3 \) map is an 8-degree polynomial, a bit complex to study.
Intersections between the graph and the diagonal correspond to the solutions of $f^3(x) = x$

Only the black dots correspond to fixed points, and there are 3 of them, corresponding to the 3-period cycle

The slope of the function, $|f'|$ is greater than 1 for the white dots, and less than 1 for the black ones

For the other intersections, they correspond to fixed points or 1-period

$r = 3.835$, inside 3-period window
The 6 intersections of interest have vanished!
Not anymore periodic behavior
For some $r$ between 3.8 and 3.835 the graph is tangent to the diagonal
At this critical value of $r$, the stable and unstable 3-period cycles coalesce and annihilate in a tangent bifurcation, that sets the beginning of the periodic window
It can be computed analytically that this happens at $r = 1 + \sqrt{6}$
Just after the tangent bifurcation, the slope at black dots (periodic points) is $\approx +1$ (a bit less)

For increasing values of $r$, hills and valleys become steeper / deeper

The slope of $f^3$ at the black dots decreases steadily from $\approx +1$ to -1. When this occurs, a flip bifurcation happens, that causes each of the fixed periodic points to split in two

$\rightarrow$ the 3-period cycle becomes a 6-period cycle!

... the process iterates as the map iterates, bringing the period doubling cascade!