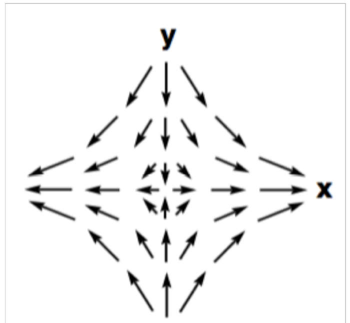
15-382 Collective Intelligence – S19

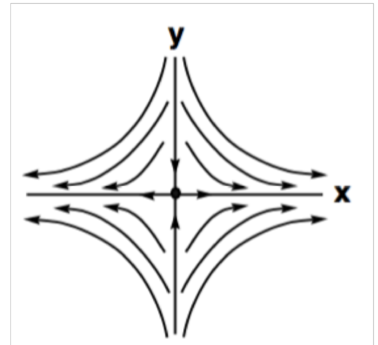
Lecture 4:
Dynamical Systems 3

Teacher:
Gianni A. Di Caro

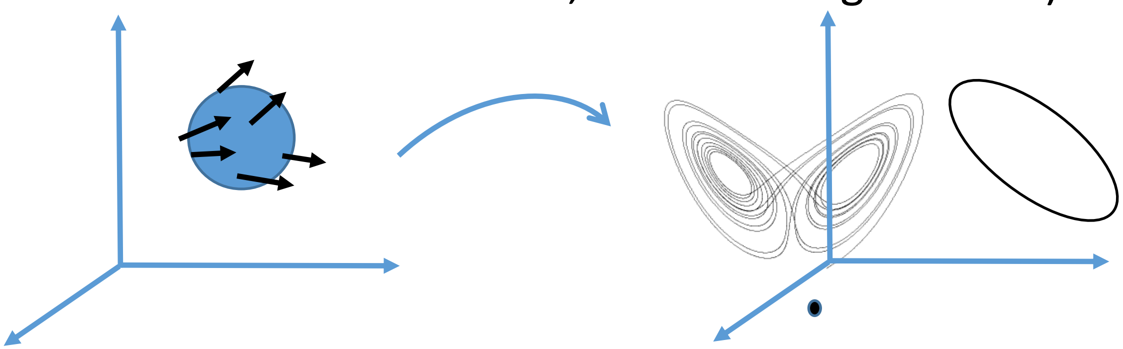
جامعة كارنيجي ميلون في قطر
Carnegie Mellon University Qatar

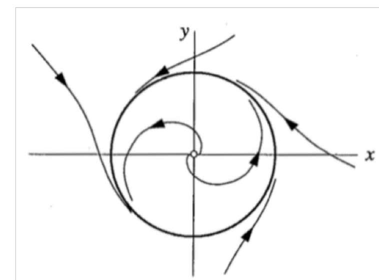
EQUILIBRIUM AND FLOWS: DISSIPATIVE VS. CONSERVATIVE SYSTEMS



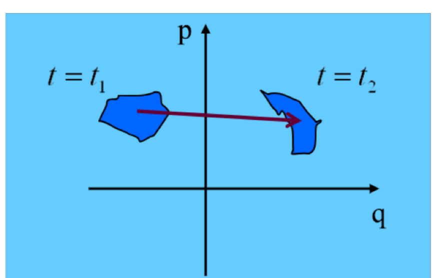


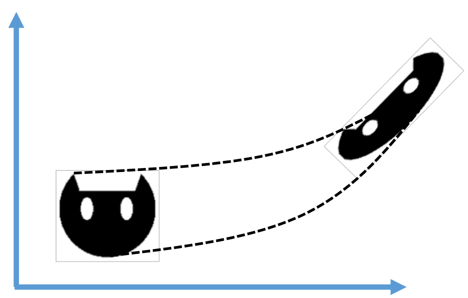
- **Dissipative system**: volumes in the phase space <u>contract</u> under the flows (reduces to set of zero measure, in the Lebesgue sense)



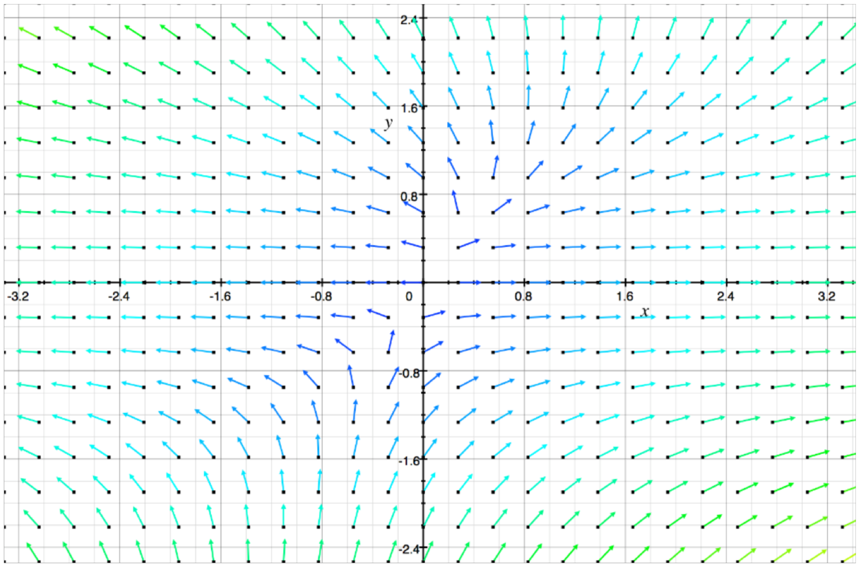


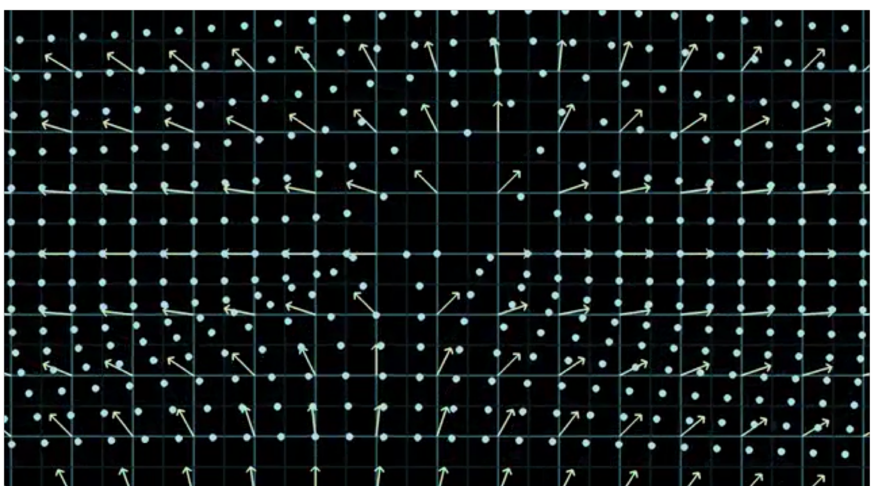
- **Conservative system**: volumes in the phase space are <u>conserved</u> under the flows





Flows and Divergence





A vector field $\boldsymbol{v}$

$$\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots$$

$$\nabla \cdot \vec{\mathbf{v}} = \overbrace{\frac{\partial v_1}{\partial x}}^{\substack{\text{Change in density} \\ \text{in the } x\text{-direction}}} + \underbrace{\frac{\partial v_2}{\partial y}}_{\substack{\text{Change in density} \\ \text{in the } y\text{-direction}}}$$

Divergence operator of the field $\boldsymbol{v}$ is a scalar function of $(x, y, \dots)$ that measures the change in a point due to the vector field

Think of a vector field in terms of the *fluid flow* that it could generate: the <u>change in density</u> of particles about a point is measured by the *divergence*

Flows and Divergence

Components of the vector-valued function $\vec{\mathbf{v}}$

$$\overbrace{\nabla}^{\text{“divergence of } \vec{\mathbf{v}}\text{”}} \cdot \vec{\mathbf{v}}(x, y, \ldots) = \frac{\partial v_1}{\partial x}(x, y, \ldots) + \frac{\partial v_2}{\partial y}(x, y, \ldots) + \cdots$$

$$\Downarrow$$

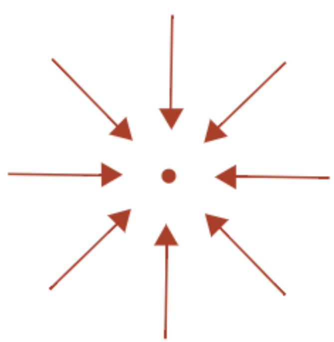
$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} v_1(x, y, \ldots) \\ v_2(x, y, \ldots) \\ \vdots \end{bmatrix}}$$

$$\operatorname{div} \vec{\mathbf{v}} = \nabla \cdot \vec{\mathbf{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \cdots$$

$$\nabla \cdot \vec{\mathbf{v}} = \overbrace{\frac{\partial v_1}{\partial x}}^{\substack{\text{Change in density} \\ \text{in the } x\text{-direction}}} + \underbrace{\frac{\partial v_2}{\partial y}}_{\substack{\text{Change in density} \\ \text{in the } y\text{-direction}}}$$

Density increase

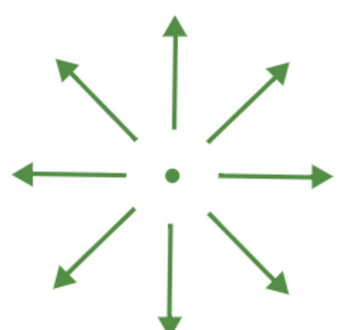
$$\nabla \cdot \vec{\mathbf{v}} < 0$$



Point is a sink

Density decrease

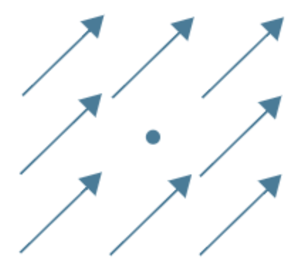
$$\nabla \cdot \vec{\mathbf{v}} > 0$$



Point is a source

Density unchanged

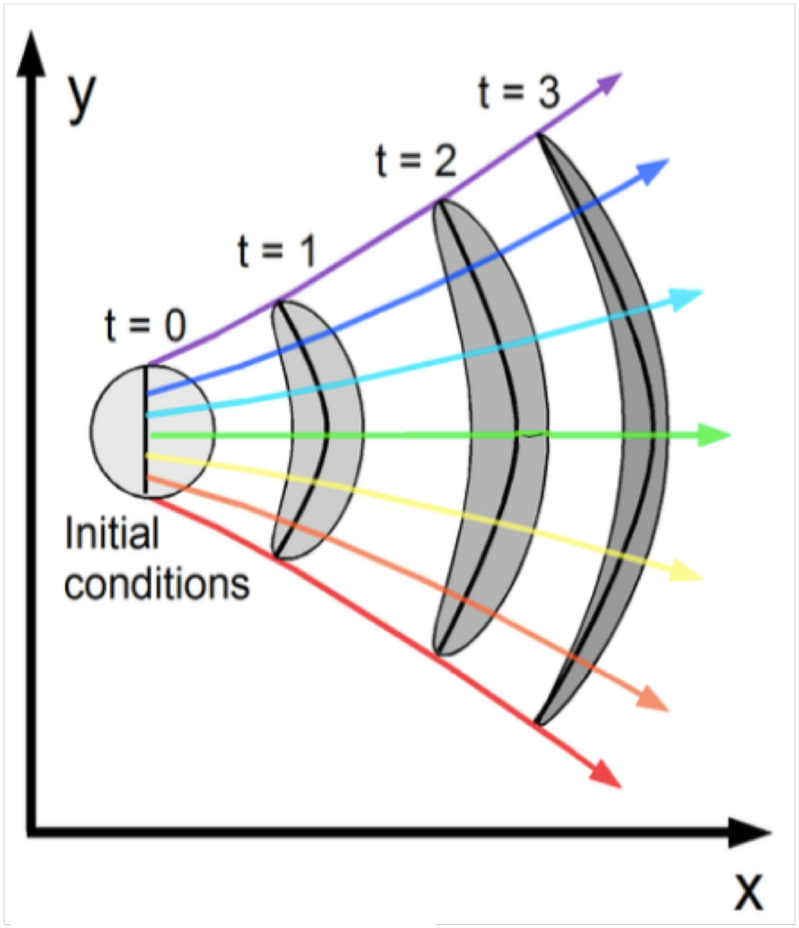
$$\nabla \cdot \vec{\mathbf{v}} = 0$$



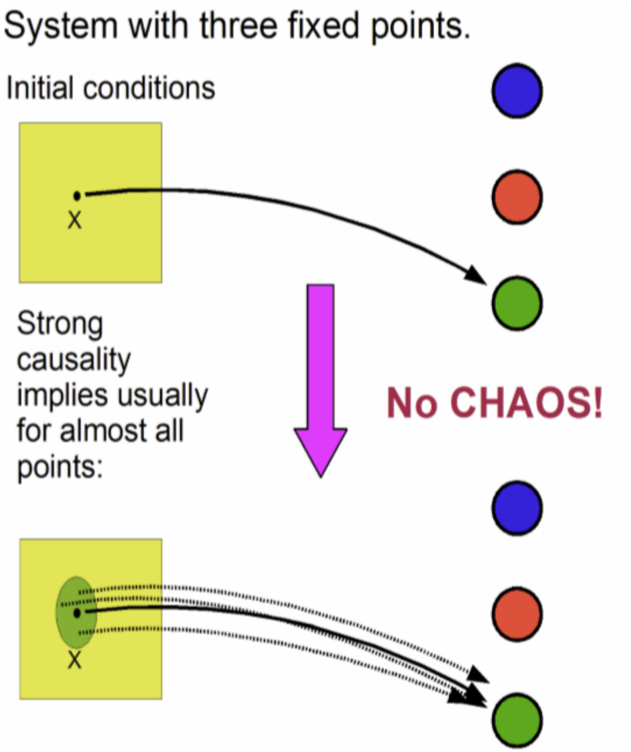
Point is a transit

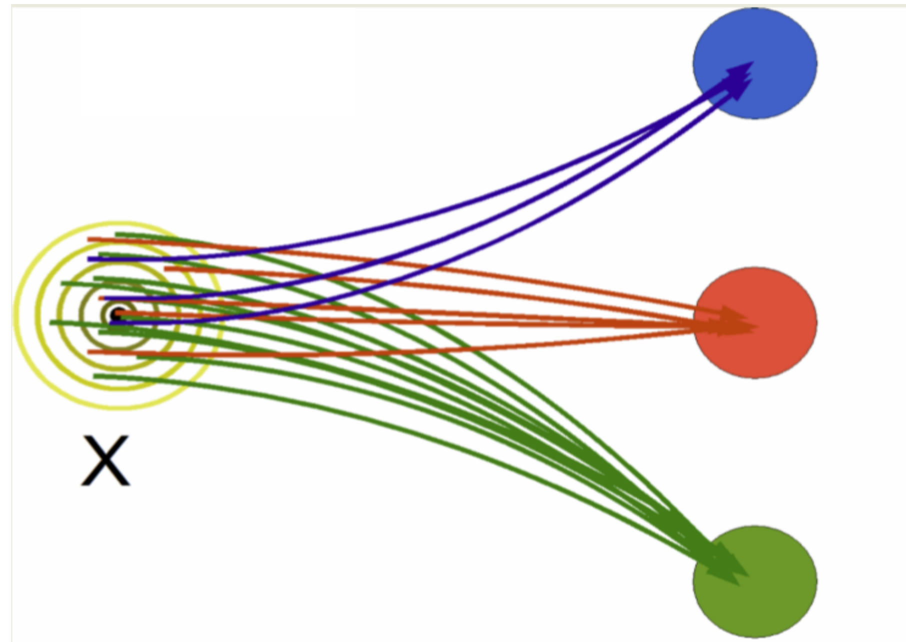
FLOWS OF INITIAL CONDITIONS IN THE PHASE SPACE

How a solid ball of initial conditions gets transformed by the flows of the dynamical system? (think about the previous analogy with solid points, with infinitely many of them, all packed in an n-dimensional ball)



REGULAR VS. CHAOTIC SCENARIO

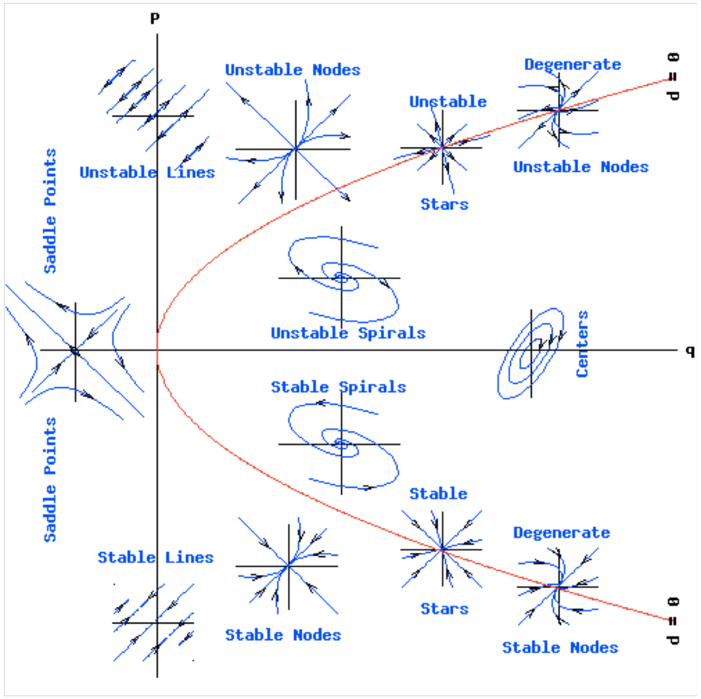


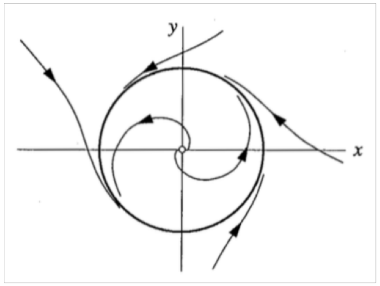


If point *X* goes to the green attractor, the same happens for all points in an open neighborhood about *X*. The volume of the initial conditions may <u>stretch or contract</u> but will not be dispersed, they will stick together

Point *X* goes to the green attractor, but the same does *not* happen for the points in its neighborhood. In the example, they end up in different attractors, but, more in general, they will end up generating different <u>aperiodic orbits</u>, *dispersing* the volume of the initial conditions but still remaining confined in some strange attractor

FIXED-POINT, PERIODIC, STRANGE *ATTRACTORS*





Up to second-order systems, $n \leq 2$

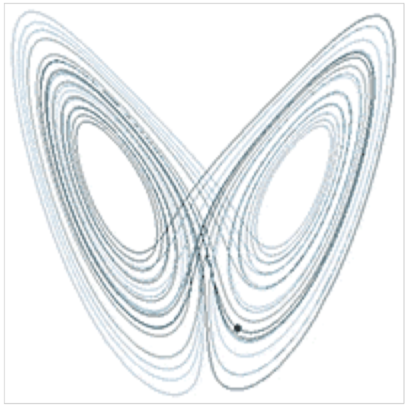
Regular attractors:

- Points (topological dimension: 0)
- Curves (topological dimension: 1)

For higher order systems, $n \geq 3$, novel geometry of attractors and complicated aperiodic dynamics can be observed

Strange attractors:

- Fractal dimension ≠ Topological dimension
- Lorenz attractor: Fractal dimension 2.06



https://en.wikipedia.org/wiki/File:A_Trajectory_Through_Phase_Space_in_a_Lorenz_Attractor.gif