Lecture 5: Dynamical Systems 4

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Volume contraction: Formalization

- Closed surface \( S(t) \) of a volume \( V(t) \) in the phase space
- (infinite) Set of initial conditions
- Let’s evolve it for \( dt \rightarrow S(t + dt) \)
- What is the volume \( V(t + dt) \)?

- \( f \) is the instantaneous velocity of the points subject to the field
- In \( dt \), a patch of area \( dA \), sweeps out a volume \((f \cdot n \, dt)dA\)

\[
V(t + dt) = V(t) + \int_S (f \cdot n \, dt) \, dA
\]

Side view of the volume
Divergence theorem in 3D: the total flux across the boundaries of a surface $S$, that in our case is $\int_S \mathbf{f} \cdot \mathbf{n} \, dA$, equals the total divergence of the vector field $\mathbf{f}$ inside the entire volume $V$ enclosed by the surface, $\int_V \nabla \cdot \mathbf{f} \, dV$

$$\dot{V} = \int_V \nabla \cdot \mathbf{f} \, dV$$

$$\nabla \cdot \mathbf{f} = \frac{\partial}{\partial x} [\sigma (y - x)] + \frac{\partial}{\partial y} [rx - y - xz] + \frac{\partial}{\partial z} [xy - bz]$$

$$= -\sigma - 1 - b < 0.$$  

$$\dot{V} = -(\sigma + 1 + b)V$$  

Volumes shrink exponential fast!

Lorenz system

$$\frac{dx}{dt} = \sigma (-x + y),$$

$$\frac{dy}{dt} = rx - y - xz,$$

$$\frac{dz}{dt} = -bz + xy.$$
A Lorenz system cannot have repelling fixed points or repelling closed orbits

- Repellers are in contradiction for volume contraction, since they are sources of volumes
- Let’s enclose a repeller with a solid surface of initial conditions nearby in the phase space
- A short time later, the surface (e.g. a sphere) will have expanded because the trajectories are driven away
- → Volume of the surface would increase and not decrease!

→ All fixed points must be sinks, or saddles
→ All closed orbits (if exists) must be stable or saddle-like
Up to second-order systems, $n \leq 2$

Regular attractors:
- Points (topological dimension: 0)
- Curves (topological dimension: 1)

For higher order systems, $n \geq 3$, novel geometry of attractors and complicated aperiodic dynamics can be observed

Strange attractors:
- Fractal dimension $\neq$ Topological dimension
- Lorenz attractor: Fractal dimension 2.06

https://en.wikipedia.org/wiki/File:A_Trajectory_Through_Phase_Space_in_a_Lorenz_Attractor.gif
A state $x^e$ is said an equilibrium state of a dynamical system $\dot{x} = f(x)$, if and only if

$$x^e = x(t; x^e; u(t) = 0), \quad \forall t \geq t_0$$

If a trajectory reaches an equilibrium state (and no input $u$ is applied) the trajectory will stay at the equilibrium state forever: internal system’s dynamics doesn’t move the system away from the equilibrium point, velocity is null: $f(x^e) = 0$
When a displacement (a force) is applied to an equilibrium condition:

- **Stable equilibrium**
- **Unstable equilibrium**
- **Neutral equilibrium**

**Metastable equilibrium**

- Why are equilibrium properties so important?
- For the same definition of an abstract model of a (complex) real-world scenario
Abelian sandpile model (starting with one billion grains pile in the center)
**Lyapunouv vs. Structural equilibrium**

- **Structural equilibrium**: is the equilibrium persistent to (small) variations in the *structure* of the systems? → Sensitivity to the value of the parameters of the vector field \( f \)

- **Lyapunouv equilibrium**: stability of an equilibrium with respect to a small deviation from the equilibrium point
**IS THE EQUILIBRIUM (LYAPUNOUV) STABLE?**

- An equilibrium state $x^e$ is said to be **Lyapunouv stable** if and only if for any $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$ such that the inequality
  \[
  \|x(0) - x^e\| \leq \delta
  \]
  
implies that $\|x(t; x(0), u(t) = 0) - x^e\| \leq \varepsilon \quad \forall t \geq 0$

- An equilibrium state $x^e$ is stable (in the Lyapunouv sense) if the response following after starting at any initial state $x(0)$ that is sufficiently near $x^e$ will not move the state far away from $x^e$
IS THE EQUILIBRIUM (LYAPUNOVA) STABLE?

What is the difference between a stable and an asymptotically stable equilibrium?
If an equilibrium state $x^e$ is **Lyapunov stable** and every motion starting sufficiently near to $x^e$ converges (goes back) to $x^e$ as $t \to \infty$, the equilibrium is said **asymptotically stable**

$\varepsilon, \delta(\varepsilon) \to 0$ as $t \to \infty$
**Informally**: a set to which all neighboring trajectories converge

**Attractor:**
- A closed set $A$
- $A$ is an invariant set: any trajectory $x(t)$ that starts in $A$ stays in $A$
- $A$ attracts, as $t \to \infty$ an open set of initial conditions: there is an open set $U$ that contains $A$, such that, if $x(0) \in U$, $x(t)$ tends to $A$ as $t \to \infty$. $A$ attracts an open set of initial conditions that starts near $A$. The largest set $U$ is $A$’s basin of attraction
- $A$ is minimal: there’s no proper subset of $A$ that satisfies previous properties

Stable fixed points

Stable limit cycles
Is \( I = \{ -1 \leq x \leq 1, y = 0 \} \) an attractor?

- Closed set

- \( A \) is an invariant set

- As \( t \to \infty \), it attracts an open set of initial conditions:

- Is minimal 😞. No, the fixed points \((\pm 1,0)\) are inside the closed set \( I \). But actually they are the only attractors for the system
Strange Attractor:

- An attractor that exhibits **sensitive dependence on initial conditions**
  - Two initial conditions in the set $U$ that are arbitrarily close at $t = 0$, become far significantly far apart as $t$ grows over time, but still remain confined in the set that defines the attractor

- Geometrically: Has **fractal** dimension

- **Deterministically chaotic attractors**
SOLUTION OF LINEAR ODEs

- The general form for an ODE: \( \dot{x} = f(x) \), where \( f \) is a \( n \)-dim vector field.

- The general form for a linear ODE (of the first order and homogeneous):
  \[
  \dot{x} = Ax, \quad x \in \mathbb{R}^n, \quad A \text{ an } n \times n \text{ coefficient matrix}
  \]

  \[
  \begin{align*}
  \dot{x}_1 &= -4x_1 - 3x_2 \\
  \dot{x}_2 &= 2x_1 + 3x_2
  \end{align*}
  \]

  \[
  A = \begin{pmatrix} -4 & -3 \\ 2 & 3 \end{pmatrix}
  \]

- **First order**: only derivatives of the first order are present (in any case, an \( n \)-order ODE can be transformed into a system of \( n \) first order ODEs, such that considering systems of first order equations is not restrictive.

- **Homogeneous**: \( \dot{x} = Ax \)

- **Non-homogeneous**: \( \dot{x} = Ax + C \), meaning that a constant term \( C \) is present in all equations (e.g., \( \dot{x}_1 = -4x_1 - 3x_2 + 1 \)).
A solution is a differentiable function $x(t)$ that satisfies the vector field.

A solution is a vector function with $n$ function components $x_i(t)$, one for each dimensional variable $x_i, i = 1, \cdots, n$.

Because of the linearity, each $x_i(t)$ takes the form of exponentials (or sum of exponentials) as we have seen for the case of population growth.

E.g., for the two-dimensional system above:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$x_1(t) = -\frac{4}{5} e^{2t} + \frac{9}{5} e^{-3t}$$
$$x_2(t) = \frac{8}{5} e^{2t} - \frac{3}{5} e^{-3t}$$

$$x(t) = -\frac{4}{5} e^{2t} \mathbf{u}_1 + \frac{3}{5} e^{-3t} \mathbf{u}_2$$

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$\mathbf{u}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$
Theorem: Linear combination of solutions of a linear ODE
If the vector functions \( x^{(1)} \) and \( x^{(2)} \) are solutions of the linear system \( \dot{x} = Ax \), then the linear combination \( c_1 x^{(1)} + c_2 x^{(2)} \) is also a solution for any real constants \( c_1 \) and \( c_2 \).

E.g., for the two-dimensional system used before, two solutions are:

\[
\begin{align*}
x^{(1)} &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -2 \ e^{2t} \end{pmatrix} \\
x^{(2)} &= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{-3t} \\ -e^{-3t} \end{pmatrix}
\end{align*}
\]

\[
c_1 x^{(1)} + c_2 x^{(2)} = \begin{pmatrix} c_1 e^{2t} + 3c_2 e^{-3t} \\ -2c_1 e^{2t} - c_2 e^{-3t} \end{pmatrix}
\]

Corollary: Any linear combination of solutions is a solution
By repeatedly applying the result of the theorem, it can be seen that every finite linear combination \( x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + \ldots + c_k x^{(k)}(t) \) of solutions \( x^{(1)}, x^{(2)}, \ldots, x^{(k)} \) is itself a solution to \( \dot{x} = Ax \).
**Theorem:** Linearly independent solutions

If the vector functions \( x^{(1)}, \ x^{(2)}, \ldots, x^{(n)} \) are linearly independent solutions of the \( n \)-dim linear system \( \dot{x} = Ax \) (for each point in the time domain), then, each solution \( x(t) \) can be expressed uniquely in the form:

\[
x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + \ldots c_n x^{(n)}(t)
\]

- This says that the solution space of a linear ODE is a **vector space**, whose *base* is defined by a set of linearly independent solutions

**Corollary:** Fundamental and general solution of a linear system

If solutions \( x^{(1)}, \ x^{(2)}, \ldots, x^{(n)} \) are linearly independent, they are the *fundamental solutions* on the domain (base of the vector space), and the *general solution* to a linear \( \dot{x} = Ax \), is given by:

\[
x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + \ldots c_n x^{(n)}(t)
\]
RECAP ON LINEAR INDEPENDENCE

- **Theorem:** Linear independence of $n$ vectors
  Given $n$ vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ of a vector space $X$, the vectors are linearly independent iff the only choice of the real coefficients $c_i$ that makes the equality below hold is $c_i = 0$ for all $i = 1, \ldots, n$
  \[ c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} + \cdots + c_n \mathbf{x}^{(n)} = 0 \]

- Linear independence means that none of the vectors can be derived from the remaining $n - 1$ as a linear combination

- E.g., in a 2-dim space: $c_1 \mathbf{x}^{(1)} + c_2 \mathbf{x}^{(2)} = 0$
  $\rightarrow \mathbf{x}^{(1)} = -\frac{c_2}{c_1} \mathbf{x}^{(2)}$ or $\mathbf{x}^{(2)} = -\frac{c_1}{c_2} \mathbf{x}^{(1)}$ if we can find a choice of $c_1, c_2 \neq 0$ that satisfies the equations, it means that one vector can be expressed as a linear function of the other (as the other times a coefficient), that is, the two vectors are *linearly dependent.*

  If no choice of $c_1, c_2 \neq 0$ can satisfy the equations, than the two vectors are *linearly independent* 

- A function space, such as the solution space of ODEs can define a vector space, and the theorem would hold in the same form
Corollary: Non-null Wronskian as condition for linear independence

The proof of the theorem on linear independence for ODE solutions uses the fact that if \( x^{(1)}, x^{(2)}, \ldots, x^{(n)} \) are linearly independent (on the time domain), then \( \det x(t) \neq 0 \)

\[
x(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}
\]

Wronskian = \( W(t) = \det x(t) \)

Therefore, \( x^{(1)}, x^{(2)}, \ldots, x^{(n)} \) are linearly independent if and only if

\[ W[x^{(1)}, x^{(2)}, \ldots, x^{(n)}](t) \neq 0 \]

E.g., for the system considered before:

\[
x^{(1)} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -2 e^{2t} \end{pmatrix} \quad x^{(2)} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3 e^{-3t} \\ - e^{-3t} \end{pmatrix}
\]

\[ W(t) = \det \begin{pmatrix} e^{2t} & 3 e^{-3t} \\ -2 e^{2t} & - e^{-3t} \end{pmatrix} = -e^{2t} e^{-3t} + 6 e^{2t} e^{-3t} = 5 e^{-t} \neq 0 \quad \forall t \]

Note: The Wroskian, expressed in a more general form, is used to assess linear independence in generic functional vector spaces (see next slide)
Consider a set of $n$ continuous functions $y_i(x) \ [i = 1, 2, 3, \ldots, n]$, each of which is differentiable at least $n$ times. Then if there exist a set of constants $\lambda_i$ that are not all zero such that

$$
\lambda_i y_1(x) + \lambda_2 y_2(x) + \cdots + \lambda_n y_n(x) = 0, \tag{1}
$$

then we say that the set of functions \{\(y_i(x)\)\} are linearly dependent. If the only solution to eq. (1) is $\lambda_i = 0$ for all $i$, then the set of functions \{\(y_i(x)\)\} are linearly independent.

The Wronskian matrix is defined as:

$$
\Phi[y_i(x)] = \begin{pmatrix}
  y_1 & y_2 & \cdots & y_n \\
  y'_1 & y'_2 & \cdots & y'_n \\
  y''_1 & y''_2 & \cdots & y''_n \\
  \vdots & \vdots & \ddots & \vdots \\
  y^{(n-1)}_1 & y^{(n-1)}_2 & \cdots & y^{(n-1)}_n
\end{pmatrix},
$$

where

$$
y'_i \equiv \frac{dy_i}{dx}, \quad y''_i \equiv \frac{d^2y_i}{dx^2}, \quad \cdots, \quad y^{(n-1)}_i \equiv \frac{d^{(n-1)}y_i}{dx^{(n-1)}}.
$$

The Wronskian is defined to be the determinant of the Wronskian matrix,

$$
W(x) \equiv \det \Phi[y_i(x)]. \tag{2}
$$

According to the contrapositive of eq. (8.5) on p. 133 of Boas, if \{\(y_i(x)\)\} is a linearly dependent set of functions then the Wronskian must vanish. However, the converse is not necessarily true, as one can find cases in which the Wronskian vanishes without the functions being linearly dependent. (For further details, see problem 3.8–16 on p. 136 of Boas.)

Nevertheless, if the $y_i(x)$ are solutions to an $n$th order ordinary linear differential equation, then the converse does hold. That is, if the $y_i(x)$ are solutions to an $n$th order ordinary linear differential equation and the Wronskian of the $y_i(x)$ vanishes, then \{\(y_i(x)\)\} is a linearly dependent set of functions. Moreover, if the Wronskian does not vanish for some value of $x$, then it is does not vanish for all values of $x$, in which case an arbitrary linear combination of the $y_i(x)$ constitutes the most general solution to the $n$th order ordinary linear differential equation.
General and Unique Solutions for Linear ODEs

- **Theorem:** Use of the Wronskian to check fundamental solutions
  If \( x^{(1)}, x^{(2)}, \ldots, x^{(n)} \) are solutions, then the Wronskian is either identically to zero or else is never zero for all \( t \) (corollary of Abel’s theorem)

- **Corollary:** To determine whether a given set of solutions are fundamental solutions it suffices to evaluate \( W[x^{(1)}, x^{(2)}, \ldots, x^{(n)}](t) \) at any point \( t \)

- **Theorem:** Existence and uniqueness of the solution
  If \( x^{(1)}, x^{(2)}, \ldots, x^{(n)} \) are solutions and the Wronskian is nonzero, then these are the fundamental solutions of the system, and the general solution of the system is in the form:
  \[
  x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + \ldots + c_n x^{(n)}(t).
  \]
  If also the initial value \( x(0) = x_0 \) is given, then there exists a unique solution for all \( t \in \mathbb{R} \).