15-382 Collective Intelligence – S19

Lecture 9:
Dynamical Systems 8

Teacher:
Gianni A. Di Caro

جامعة كارنيجي ميلون في قطر
Carnegie Mellon University Qatar

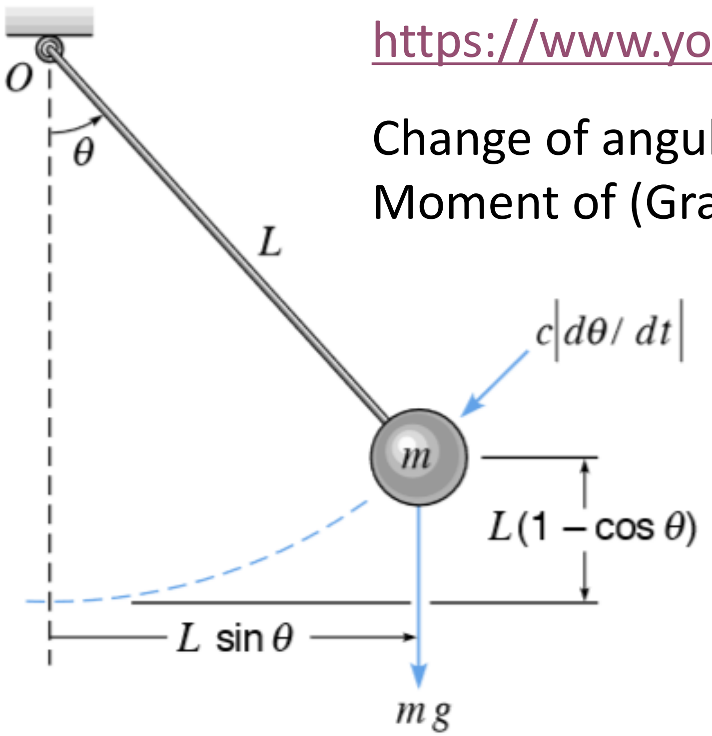
RESULTS FROM LINEARIZATION

- Theorem (Stability of critical points of non-linear systems):
  - Let r_1 and r_2 be the eigenvalues of the linear system $\dot{\boldsymbol{x}} \approx J_{\boldsymbol{f}}(\boldsymbol{x}_0)\boldsymbol{x} = A\boldsymbol{x}$ resulting from the linearization of an original non-linear system about the critical point $\boldsymbol{x} = \boldsymbol{0}$ (via the definition of a quasi-linear system)
  - The type and stability of the critical point on the linear and non linear system are the following:

	Linear System		Almost Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

EXAMPLE: DAMPED PENDULUM



https://www.youtube.com/watch?v=oWiuSp6qAPk

Change of angular momentum about the origin = Moment of (Gravitational force + Damping force)

$$mL^2\frac{d^2\theta}{dt^2} = -cL\frac{d\theta}{dt} - mgL\sin\theta.$$

$$\frac{d^2\theta}{dt^2} + \frac{c}{mL}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = 0,$$

$$\boxed{\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\sin\theta = 0,}$$

$$\gamma = c/mL \text{ and } \omega^2 = g/L$$

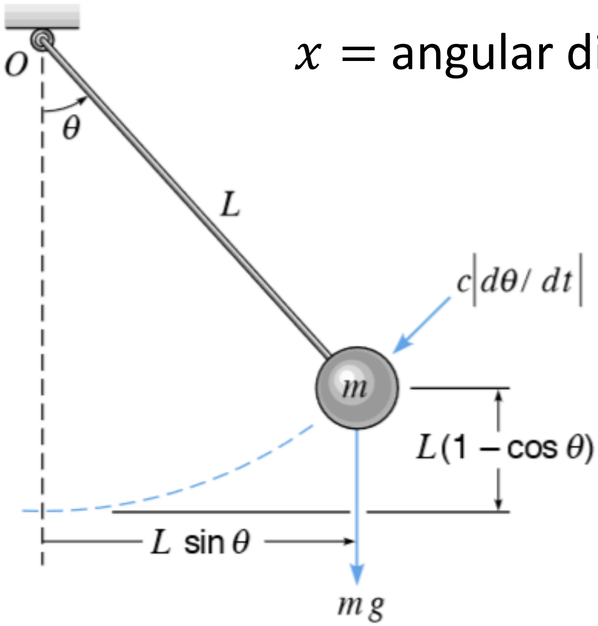
Second order ODE → Convert to a system of two 1st order equations:

$$x = \theta \text{ and } y = d\theta/dt$$

$$\boxed{\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2\sin x - \gamma y.}$$

EXAMPLE: DAMPED PENDULUM

$$x = \theta \text{ and } y = d\theta/dt$$

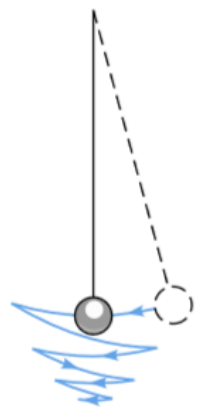


x = angular displacement θ $\quad$ y = velocity of angular displacement θ

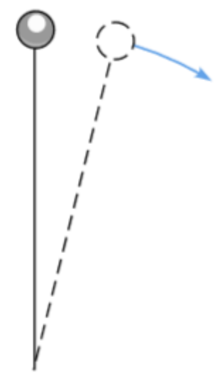
$$\boxed{\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y.}$$

Critical points: (where θ's rate of change, y, is zero)

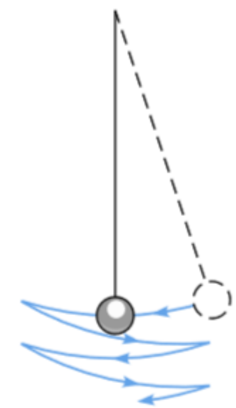
$$y = 0 \text{ and } x = \pm n\pi$$



$\theta = 0$
Stable



$\theta = \pi$
Unstable



$\gamma(c) = 0$
oscillatory

EXAMPLE: DAMPED PENDULUM

Critical points

$y = 0$ and $x = \pm n\pi$

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\omega^2 \sin x - \gamma y.$$

$$J_f = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -\gamma \end{pmatrix}$$

$(0,0)$

$$\begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}$$

$(\pi, 0)$

$$\begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}$$

- In $(0,0)$: $r_1, r_2 = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4\omega^2}}{2}$ $\qquad \sqrt{\gamma^2 - 4\omega^2} < \gamma$

- $\gamma^2 - 4\omega^2 > 0$: Damping is strong, eigenvalues are real, unequal, negative → (0,0) is an asymptotically stable node of the linear, as well as, non linear system

- $\gamma^2 - 4\omega^2 = 0$: Eigenvalues are real, equal, negative → (0,0) is an asymptotically stable (proper or improper) node of the linear system. It may be either an asymptotically stable node or spiral point of the non linear system

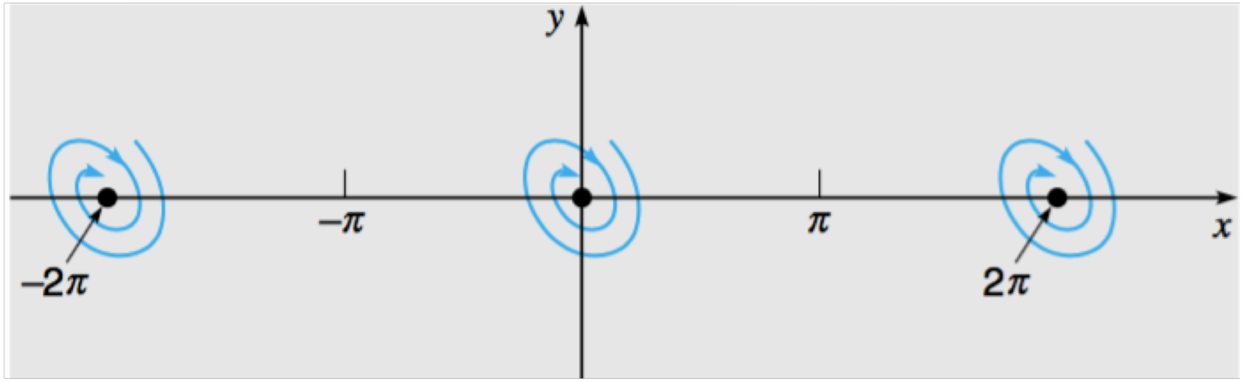
- $\gamma^2 - 4\omega^2 < 0$: Damping is weak, eigenvalues are complex with negative real part. → (0,0) is an asymptotically stable spiral of both linear and non linear systems

DAMPED PENDULUM

- → $(0,0)$ is a spiral point of the system if the damping is small and a node if the damping is large enough. In either case, the origin is *asymptotically stable.*

- Can we directly derive the **direction of motion** on the spirals near (0,0) with small damping, $\gamma^2 - 4\omega^2 < 0$?
  - Being a spiral about the origin, the trajectory will intersect the positive y-axis ($x = 0$ and $y > 0$). At such a point, from equation $dx/dt = y > 0$ it follows that the x-velocity is positive, meaning that the direction of motion is clockwise (analogously, we could say that also the point $(0, y < 0)$ is an intersection, and from $dx/dt = y < 0$ we get that in the two $y < 0$ quadrants , because of $dx/dt = y < 0$, the motion is counterclockwise)

- Equilibrium at points $(\pm n\pi, 0)$ with n even is the same as in $(0{,}0)$, these are in fact all corresponding to the same configuration of the downward equilibrium position of the pendulum



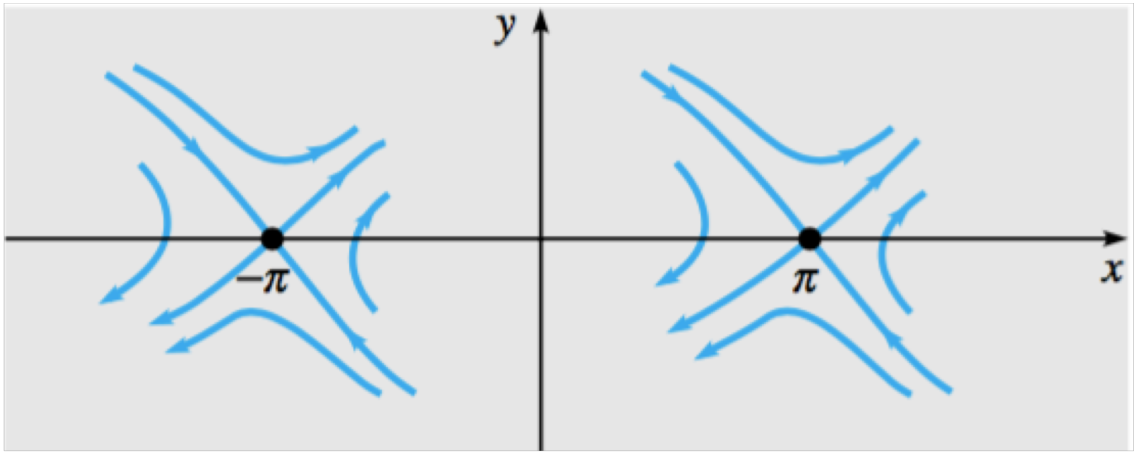
DAMPED PENDULUM

- At equilibrium point $(\pi, 0)$, the eigenvalues of the Jacobian are:

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 + 4\omega^2}}{2}$$

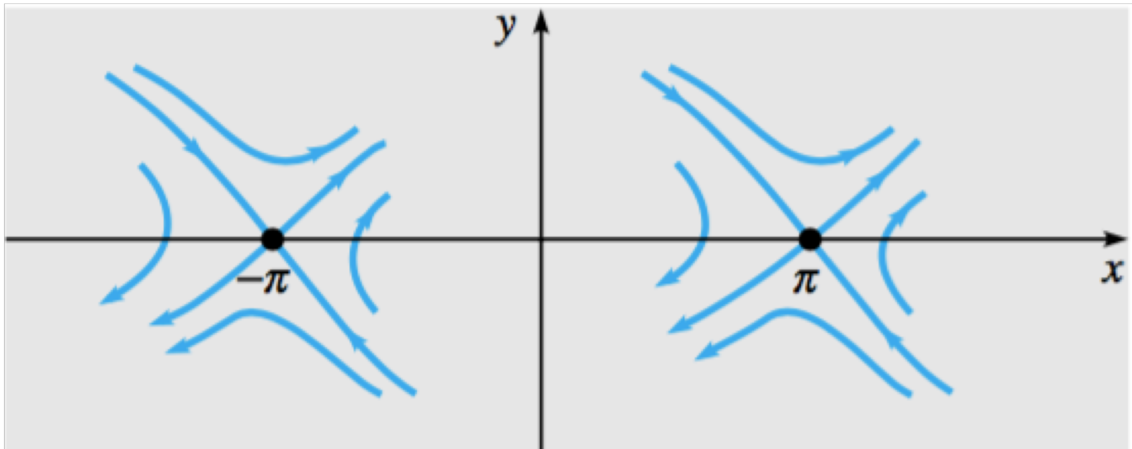
Since $\gamma^2 + 4\omega^2 > \gamma$, the eigenvalues always have opposite sign, $r_1 > 0, r_2 < 0$, making the point a **saddle** → Regardless of the damping, the equilibrium is an <u>unstable</u> saddle (for both the linear and the original system)

- The same applies to all other equilibrium points $(n\pi, 0)$, with n <u>odd</u>



- How do we derive the direction of motion near the equilibrium?

Damped Pendulum



- General linearized solution near the equilibrium: $$\begin{pmatrix} u \\ v \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ r_1 \end{pmatrix} e^{r_1 t} + C_2 \begin{pmatrix} 1 \\ r_2 \end{pmatrix} e^{r_2 t}$$

- Because of $r_1 > 0, r_2 < 0$, the linearized solution that approaches zero (i.e., that approaches equilibrium point) as $t \to \infty$, must correspond to $C_1 = 0$ (otherwise either u and/or v would grow exponentially)

- For this solution, the slope of entering trajectories is $\frac{v}{u} = r_2 < 0$, one lies in the first quadrant, the other in the fourth, as shown in the figure

- The pair of (linearized) trajectories exiting from the saddle point correspond to $C_2 = 0$, that have a constant slope $r_1 > 0$, and lies in 1st and 4th quadrant

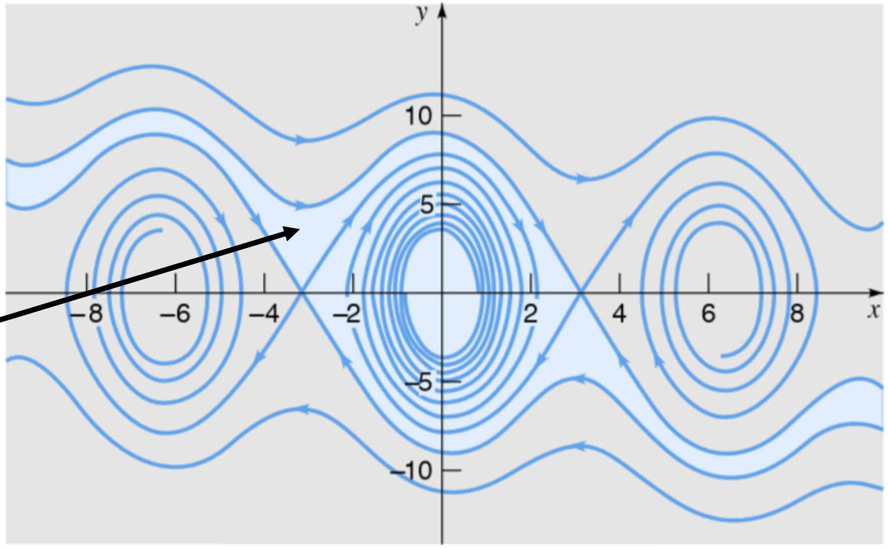
BASIN OF ATTRACTION: A GLOBAL NOTION

An instance of the pendulum model: $dx/dt = y, \qquad dy/dt = -9\sin x - \frac{1}{5}y$

- The trajectories that enter the saddle points separate the phase plane into regions. Such a trajectory is called a **separatrix**. Each region contains exactly one of the asymptotically stable spiral points.

Separatrix

Basin of attraction



- **Basin of attraction:** The set of all initial points from which trajectories approach a given asymptotically stable critical point (region of asymptotic stability for that a critical point)

- Each asymptotically stable critical point has its own basin of attraction, which is bounded by the separatrices through the neighboring unstable saddle points.

NULLCLINES

- **Isoline:** set of points where a function takes same value →
  $\{(x_1, x_2) \mid H(x_1, x_2) = c\}$
- **Isocline:** set of points where a function has the same slope, along some given coordinate directions → $\{(x_1, x_2) \mid \frac{\partial H(x_1,x_2)}{\partial x_i} = c\}$
- For a differential equation of the form $\dot{x} = f(x)$, an isocline corresponds to the set $\{x \mid f(x) = c\}$ (the slope / derivative is constant) → An isocline is an *isoline* of the vector field $f(x)$
- **Nullcline:** set of points where a function has the same, null, slope.
  For a $\dot{x} = f(x)$, a nullcline corresponds to the set $\{x \mid f(x) = 0\}$
- For a system of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, a nullcline is considered with respect to each coordinate direction: a system of two ODEs:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}$$

has two nullclines sets, corresponding respectively to $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$
→ The nullclines are the curves where either $\dot{x}_1 = 0$, or $\dot{x}_2 = 0$

NULLCLINES

- The nullclines can help to construct the phase portrait
- Let's consider a 2x2 system, the following properties hold:
  - The nullclines cross at the critical / equilibrium points
  - Trajectories cross vertically the nullcline $f_1(x_1, x_2) = 0$, since for this nullcline, $\dot{x}_1 = 0$, all flow variations happen along x_2
  - Trajectories cross horizontally the nullcline $f_2(x_1, x_2) = 0$, since for this nullcline, $\dot{x}_2 = 0$, all flows variations happen along x_1
  - In regions enclosed by the nullclines, the ratio $\frac{dx_1}{dx_2}$ has constant sign: trajectories are either going upward to downward
  - Trajectories can only go flat or vertical across nullclines

NULLCLINES FOR A SIMPLE TWO POPULATIONS MODEL

$$\begin{aligned} dN_1/dt &= F_1(N_1,N_2) = N_2 \\ dN_2/dt &= F_2(N_1,N_2) = -N_1 \end{aligned}$$

Nullclines: $N_2 = 0,\ N_1 = 0$

Solution: $\dfrac{dN_1}{dN_2} = -\dfrac{N_2}{N_1} = H(N_1, N_2)$ → Variable separation and integration:

$$N_1(t)^2 + N_2(t)^2 = \text{constant} = N_{10}^2 + N_{20}^2$$