

Chapter 6: Numerical Methods

6.1 Euler Method

Basic Idea

- ODE: $y' = f(t, y)$
- Assume $y(t)$ is known
- For small h approximate

$$\frac{y(t+h) - y(t)}{h} \approx y'(t) = f(t, y(t))$$

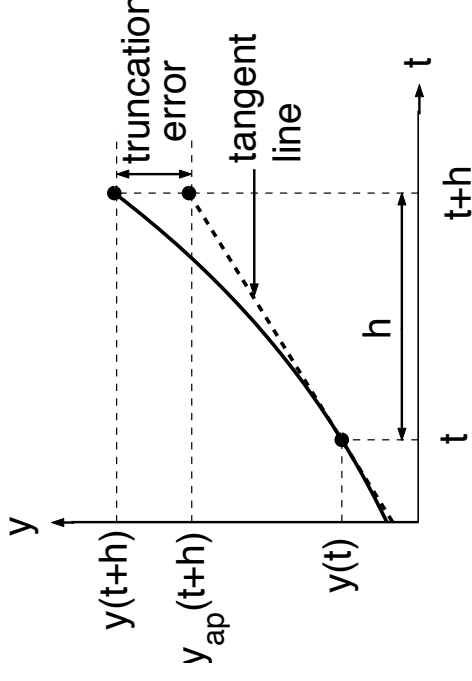
$$\Rightarrow y(t+h) \approx y_{ap}(t+h)$$

where

$$y_{ap}(t+h) = y(t) + hf(t, y(t))$$

- **Truncation Error:**

$$|y(t+h) - y_{ap}(t+h)|$$



Iteration Scheme

IVP: $y' = f(t, y), y(t_0) = y_0$

Approximate $y(t_k) \approx y_k$ at t_k :

$$y_1 = y_0 + hf(t_0, y_0), \quad t_1 = t_0 + h$$

$$y_2 = y_1 + hf(t_1, y_1), \quad t_2 = t_1 + h$$

⋮

$$y_{k+1} = y_k + hf(t_k, y_k)$$

$$t_{k+1} = t_k + h$$

Ex: Approximate the solution to

$$y' = y, \quad y(0) = 1$$

in $0 \leq t \leq 1$. Start: $t_0 = 0, y_0 = 1$

h = 1

$$y_1 = y_0 + h f(0, 1) = 1 + 1 \cdot 1 = 2$$

$$t_1 = t_0 + h = 0 + 1 = 1$$

h = 0.5

$$y_1 = 1 + 0.5 \cdot 1 = 1.5$$

$$t_1 = 0 + 0.5 = 0.5$$

$$y_2 = 1.5 + 0.5 \cdot 1.5 = 2.25$$

$$t_2 = 0.5 + 0.5 = 1$$

h = 0.25

$$y_1 = 1 + 0.25 \cdot 1 = 1.25$$

$$t_1 = 0 + 0.25 = 0.25$$

$$y_2 = 1.25 + 0.25 \cdot 1.25 = 1.5625$$

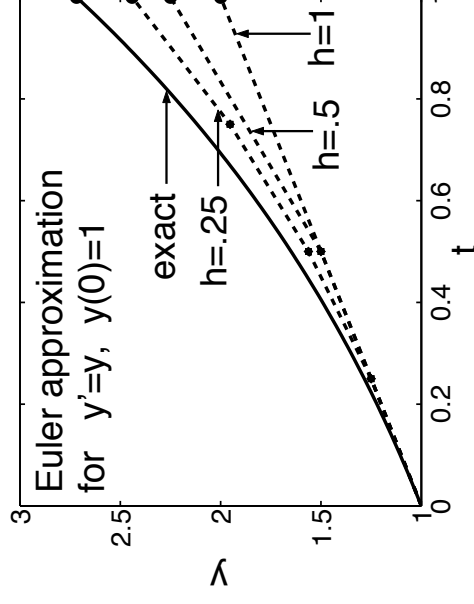
$$t_2 = 0.25 + 0.25 = 0.5$$

$$y_3 = 1.5625 + 0.25 \cdot 1.5625 = 1.953125$$

$$t_3 = 0.5 + 0.25 = 0.75$$

$$y_4 = 1.953125 + 0.25 \cdot 1.953125 = 2.44140625$$

$$t_4 = 0.75 + 0.25 = 1$$



Ex: Approximate the solution to

$$y' = t - y, \quad y(0) = 0.5$$

in $0 \leq t \leq 1$ using $h = 0.25$

Start: $y_0 = 0.5, t_0 = 0$

$$y_1 = 0.5 + 0.25 \cdot (0 - 0.5) = 0.375$$

$$t_1 = 0 + 0.25 = 0.25$$

$$y_2 = 0.375 + 0.25 \cdot (0.25 - 0.375)$$

$$= 0.3438$$

$$t_2 = 0.25 + 0.25 = 0.5$$

$$y_3 = 0.3438 + 0.25 \cdot (0.5 - 0.3438)$$

$$= 0.3828$$

$$t_3 = 0.5 + 0.25 = 0.75$$

$$y_4 = 0.3828 + 0.25 \cdot (0.75 - 0.3828)$$

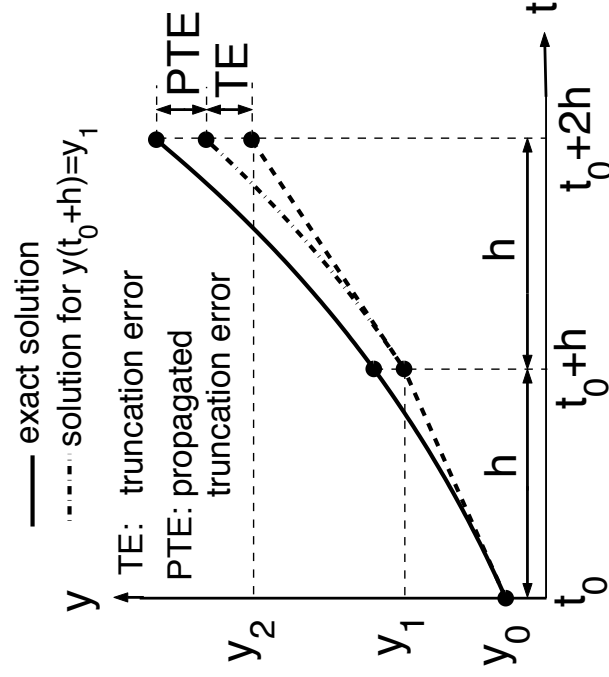
$$= 0.4746$$

$$t_4 = 0.75 + 0.25 = 1$$

Errors

Three error sources:

- Truncation error at each Euler step
- Propagated (accumulated) truncation error
- Roundoff error (not controllable)



Ex.: $y' = t - y$, $y(0) = .5$

Approximate $y(1)$ for step sizes

$h = 1/m$, $m = 1, 2, 4, 8, 16, 32$

Exact Value: $y(1) = 0.5518$

Error: $E(h) = |y(1) - y_m|$

h	y_m	$E(h)$
1	0	0.5518
1/2	0.375	0.1768
1/4	0.4746	0.0772
1/8	0.5154	0.0364
1/16	0.5341	0.0177
1/32	0.5431	0.0087

$$E(h/2) \approx E(h)/2 \Rightarrow E(h) \approx Ch$$

Theorem: There $\exists C > 0$ s.t.

$$E(h) \leq Ch$$

(Euler method is first order method)

Worked out Examples from Exercises

Ex. 1: $y' = ty$, $y(0) = 1$.

Compute five Euler-iterates for $h = 0.1$.
Arrange computation and results in a table.

k	t_k	y_k	$f(t_k, y_k) = t_k y_k$	h	$f(t_k, z_k)h$
0	0	1	0	0.1	0
1	0.1	1	0.1000	0.1	0.0100
2	0.2	1.0100	0.2020	0.1	0.0202
3	0.3	1.0302	0.3091	0.1	0.0309
4	0.4	1.0611	0.4244	0.1	0.0424
5	0.5	1.1036	0.5518	0.1	0.0552

Ex. 7: $y' + 2xy = x$, $y(0) = 8$

- (i) Compute Euler-approximations in $0 \leq x \leq 1$ for $h = 0.2$, $h = 0.1$, $h = 0.05$.
- (ii) Find exact solution
- (iii) Plot exact solution as curve and Euler approximations as points.

(i) In Matlab, Euler approximation for $h = 0.2$ is computed and stored in arrays $x0_2$, $y0_2$ via

```

h=0.2;
m=1/h; x=0; y=8;
xv=x; yv=y;
for k=1:m
    f=-2*x*y+x;
    y=y+h*f; yv=[yv y];
    x=x+h; xv=[xv x];
end
x0_2=xv; y0_2=yv;

```

Analogously for $h = 0.1$ and $h = 0.05$ (arrays $x0_1$, $y0_1$ and $x0_05$, $y0_05$).

(ii) Variation of Parameter:

$$y'_h = -2xy \Rightarrow$$

$$\begin{aligned}
 y_h(x) &= \exp\left(\int_0^x (-2x)dx\right) = e^{-x^2} \\
 y(x) &= y_h(x)\left(8 + \int_0^x [f(\xi)/y_h(\xi)]d\xi\right) \\
 &= 8e^{-x^2} + e^{-x^2} \int_0^x \xi e^{\xi^2} d\xi \\
 &= 8e^{-x^2} + e^{-x^2}(e^{x^2} - 1)/2 \\
 &= (15/2)e^{-x^2} + 1/2
 \end{aligned}$$

(iii) Matlab plot commands:

```

x=linspace(0,1,100);
y=1/2+15/2*exp(-x.^2);
plot(x0_2,y0_2,'ko', x0_1,y0_1,'k*', ...
      x0_05,y0_05,'k+', x,y,'k'),
xlabel('x'), ylabel('y')
axis([0 1 3.5 8])

```

Plot for Ex. 7

