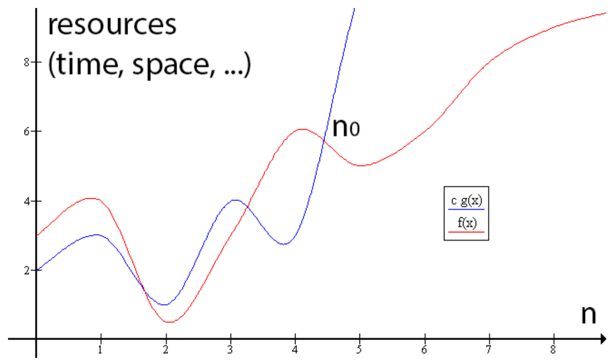
15-122: Principles of Imperative Computation — Spring 2024

Recitation 03: Function Family Reunion — Thursday January 25th

Big-O intuition



To the left of n_0, the functions can do anything.
To its right, $c\,g(n)$ is always greater than or equal to $f(n)$.

Intuitively, $O(g(n))$ is the set of all functions that $g(n)$ can outpace in the long run (with the help of a constant scaling factor c). For example, n^2 eventually outpaces $3n\log(n)+5n$, so $3n\log(n)+5n \in O(n^2)$. Because we only care about long run behavior, we generally can discard constants and can consider only the most significant term in a function.

There are actually *infinitely many functions* that are in $O(g(n))$: If $f(n) \in O(g(n))$, then $\frac{1}{2}f(n) \in O(g(n))$ and $\frac{1}{4}f(n) \in O(g(n))$ and $2f(n) \in O(g(n))$. In general, for any constants k_1, k_2, $k_1 f(n) + k_2 \in O(g(n))$.

Big-O definition

The formal definition of big-O has a lot of mathematical symbols in it, and so can be very confusing at first. Let's familiarize ourselves with the formal definition and get an intuition behind what it's saying.

$O(g(n))$ is a *set* of functions, where $f(n) \in O(g(n))$ if and only if:

there is some ______________________ and some ______________________

such that ______________________ for all ______________________.

This definition amounts to putting an asymptotic upper bound on $f(n)$.

Checkpoint 0

Using the formal definition of big-O, prove that $n^3 + 9n^2 - 7n + 2 \in O(n^3)$.

$c =$ ______________, $n_0 =$ ______________

To show: ______________________________ (expand c and n_0)

A. $n \geq$ ______________________ by assumption

B. ______________________ by ______________

C. ______________________ by ______________

D. ______________________ by ______________

E. ______________________ by ______________

F. ______________________ by ______________

Simplest, tightest bounds

Something that will come up often with big-O is the idea of a *simple* and *tight* bound on the runtime of a function.

It's technically correct to say that linear search is in $O(3n+2)$ where n is the length of the input array, but $O(3n+2)$ consists of the exact same functions as $O(n)$, which is simpler.

It's also technically correct to say that binary search, which takes around $\log n$ steps on an n-element array, is in $O(n!)$, since $n! > \log n$ for all $n > 0$ but it's not very useful. If we ask for a *tight* bound, we want the closest bound you can give. For binary search, $O(\log n)$ is a tight bound because no function that grows more slowly than $\log n$ provides a correct upper bound for binary search.

Unless we specify otherwise, we want the simplest, tightest bound!

Complexity Classes

Big-O sets in simplest and tightest form are used to summarize the complexity of a given function — for example $n^3 + 9n^2 - 7n + 2 \in O(n^3)$ highlights that $n^3 + 9n^2 - 7n + 2$ is a cubic function. As such, big-O sets in simplest and tightest form are called *complexity classes*.

When working with functions with a single argument, say n, the most common complexity classes we will encounter in this course are

$$O(1) \;\subset\; O(\log n) \;\subset\; O(n) \;\subset\; O(n \log n) \;\subset\; O(n^2) \;\subset\; O(2^n) \;\subset\; O(n!)$$

Every function in the big-O set on the left of the *subset symbol* ($\subset$) is also a function in the big-O set on the right (but not necessarily vice versa) — for example $O(\log n) \subset O(n)$ says that every function in $O(\log n)$ is also in $O(n)$.

We use big-O sets in simplest and tightest form also to classify functions with multiple arguments.

Checkpoint 1

For each of the following big-O sets, give an equivalent big-O set in simplest and tightest form.

$O(3n^{2.5}+2n^2)$ can be written more simply as ____________________

One interesting consequence of this second result is that $O(\log_i n) = O(\log_j n)$ for all i and j (as long as they're both greater than 1), because of the change of base formula:

$$\log_i n = \frac{\log_j n}{\log_j i}$$

But $\frac{1}{\log_j i}$ is just a constant! So, it doesn't matter what base we use for logarithms in big-O notation.

When we ask for the *simplest, tightest bound* in big-O, we'll usually take points off if you write, for instance, $O(\log_2 n)$ instead of the simpler $O(\log n)$.

$O(\log_{10} n + \log_2(7n))$ can be written more simply as ____________________

Checkpoint 2

Give the complexity class of the following functions:

$f(n) = 16n^2+5n+2 \in$ ____________________

$g(n,m) = n^{1.5} \times 16m \in$ ____________________

$h(x,y) = \max(x,y)+x^2 \in$ ____________________

Determining Big-O

Determine the big-O complexity of the following function.

```
1  int bigO_1(int k)
2  //@requires k >= 0;
3  {
4    int[] A = alloc_array(int, k); // allocating an k-length array takes O(k) time
5    for (int i = 0; i < k; i++) {
6      for (int j = 1; j < k; j*=2) {
7        A[i] += j;
8      }
9    }
10   int p = 0;
11   while (p < 10) {
12     f(A, k); //assume f takes O(k) time
13     p++;
14   }
15   return A[k-1];
16 }
```

Always write your complexity in terms of the input variables!

- Line 4 takes time in $O($________________$)$
- The loop on lines 5–9 runs ________________ times
  - The loop on lines 6–8 runs ________________ times
    - Each run of line 7 takes time in $O($________________$)$

    Therefore the loop on lines 6–8 takes time in $O($________________$)$

  Therefore the loop on lines 5–9 takes time in $O($________________$)$
- Line 10 takes time in $O($________________$)$
- The loop on lines 11–14 runs ________________ times
  - Each run of line 12 takes time in $O($________________$)$
  - Each run of line 13 takes time in $O($________________$)$

  Therefore the loop on lines 11–14 takes time in $O($________________$)$
- Line 15 takes time in $O($________________$)$

Thus, the function `bigO_1` takes time in $O($________________$)$ to run altogether

Checkpoint 3

Determine the big-O class of the following function. You may use the lines on the right for scratch work.

```
int bigO_2(int[] L, int m, int n)
//@requires \length(L) == m && m > 0 && n >= 0;
{
  int[] A = alloc_array(int, n);    // ____________________

  for (int i = 0; i < n; i++) {     // ____________________
    for (int j = i; j < n; j++) {   // ____________________
      A[i] = i * j;                 // ____________________
    }                               // ____________________
  }                                 // ____________________
  int c = m-1;                      // ____________________

  while (c > 0) {                   // ____________________
    L[c] += 122;                    // ____________________
    c /= 4;                         // ____________________
  }                                 // ____________________
  return L[m/2];                    // ____________________
}                                   // ____________________
```

The big-O class of this function is ______________________________.